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First Order Primal-Dual Optimization Methods

Exercise Sheet 2

Exercise 2.1 (Proximity Operator):
Solve Exercise 1.3 from Exercise Sheet 1.

Exercise 2.2 (Optimality Conditions):
In this exercise, we consider the convex optimization problem from the lecture

$$\min_{x \in C} f(x) \quad \text{s.t.} \quad Ax = b, \tag{P1}$$

with $C \subset \mathbb{R}^n$ nonempty, closed and convex, $f : U \to \mathbb{R}$ convex and continuously differentiable on an open neighborhood U of C, $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, and investigate its optimality conditions

$$\begin{aligned} &\nabla_x L(\bar{x}, \bar{\lambda})^T (x - \bar{x}) \geq 0 \qquad \forall x \in C, \\ &\bar{x} \in C, \ A\bar{x} = b, \end{aligned} \tag{OC}$$

with the Lagrange function $L(x, \lambda) = f(x) + \lambda^T (Ax - b)$ and Lagrange multiplier $\lambda \in \mathbb{R}^m$.

To that end, we first recall that $\bar{x} \in X$ is a solution of the general convex minimization problem

$$\min_{x \in X} f(x),$$

with a nonempty, convex set $X \subset \mathbb{R}^n$, and a convex, on an open neighborhood of X differentiable function $f : X \to \mathbb{R}$, if and only if $\bar{x} \in X$ and the variational inequality

$$\nabla f(\bar{x})^T (x - \bar{x}) \geq 0 \qquad \forall x \in X, \tag{VI}$$

is satisfied.

a) The **Slater condition** holds, if there exists $x_0 \in \mathbb{R}^n$ with $x_0 \in \text{int}(C)$ and $Ax_0 = b$, where $\text{int}(C)$ denotes the interior of C.

   Let now $\bar{x} \in \mathbb{R}^n$ be a solution of (P1). If additionally the Slater condition is satisfied, one can now show, that there exists $\bar{\lambda} \in \mathbb{R}^m$ such that $L(\bar{x}, \bar{\lambda})$ is a saddle point of the Lagrange function L.

   Use this result and the variational inequality (VI) for $L(\cdot, \bar{\lambda})$ to show that the conditions (OC) are met, i.e. that (OC) are *necessary optimality conditions* for (P1), if Slater's condition is satisfied.

b) Show that Slater's condition in a) is essential by constructing an example where it is violated, the minimum is given by $\bar{x}$, and for every $\lambda \in \mathbb{R}^m$, there exists $x_\lambda \in C$ such that

$$\nabla_x L(\bar{x}, \lambda)^T (x_\lambda - \bar{x}) < 0. \tag{1}$$

   Hint: Use a two dimensional example of (P1), where the convex set C is given by $C := \{x \in \mathbb{R}^2 : x_2 \geq x_1^2\}$, the equality constraint reads as $x_2 = 0$, and the objective function is given by $f(x) = x_1$. Define suitable A, b, identify the minimum $\bar{x}$, and consider the different cases $\lambda \leq 0$ and $\lambda > 0$ to find suitable x_λ satisfying (1), respectively.

Exercise 2.3 (Parametrizing minimizers of $L^a(\gamma;\cdot,\lambda)$ by λ):
Here we consider problem (P1) in the special case $C = \mathbb{R}^n$ and with $f : \mathbb{R}^n \to \mathbb{R}$ convex and twice differentiable. Recalling that for $\lambda \in \mathbb{R}^m$ and $\gamma \geq 0$, the augmented Lagrangian function $L^a(\gamma; x, \lambda)$ is given by

$$L^a(\gamma; x, \lambda) = f(x) + \lambda^T(Ax - b) + \frac{\gamma}{2}\|Ax - b\|^2,$$

we fix $\lambda = \hat{\lambda}$ and assume that $L^a(\gamma;\cdot,\hat{\lambda})$ has a minimum $\hat{x}$ at which $\nabla_x^2 L^a(\gamma;\hat{x},\hat{\lambda})$ is positive definite.

Show with the implicit function theorem that there exists an open set $U \subset \mathbb{R}^m$ such that for all $\lambda \in U$, $L^a(\gamma;\cdot,\lambda)$ has a unique minimum $x(\lambda)$, that the function $U \ni \lambda \mapsto x(\lambda)$ is differentiable, and compute its derivative.