A PRIORI ERROR ANALYSIS FOR DISCRETIZATION OF SPARSE ELLIPTIC OPTIMAL CONTROL PROBLEMS IN MEASURE SPACE

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Abstract. In this paper an optimal control problem is considered, where the control variable lies in a measure space and the state variable fulfills an elliptic equation. This formulation leads to a sparse structure of the optimal control. For this problem a finite element discretization based on [7] is discussed and a priori error estimates are derived, which significantly improve the estimates from [7]. Numerical examples for problems in two and three space dimensions illustrate our results.

Key words. optimal control, sparsity, finite elements, error estimates

AMS subject classifications.

1. Introduction. In this paper we consider the following optimal control problem:

Minimize \( J(q,u) = \frac{1}{2} \|u - u_d\|_{L^2(\Omega)}^2 + \alpha \|q\|_{\mathcal{M}(\Omega)}, \quad q \in \mathcal{M}(\Omega) \) (1.1)

subject to

\[
\begin{align*}
-\Delta u = q & \quad \text{in } \Omega \\
u = 0 & \quad \text{on } \partial \Omega.
\end{align*}
\] (1.2)

Here, \( \Omega \subset \mathbb{R}^d \) (\( d = 2, 3 \)) is a convex bounded domain with a \( C^{2,\beta} \)-boundary \( \partial \Omega \). The control variable \( q \) is searched for in the space of regular Borel measures \( \mathcal{M}(\Omega) \), which is identified with the dual of the space of continuous functions vanishing on the boundary \( C_0(\Omega) \). The state variable \( u \) is the solution of the state equation (1.2), see the next section for the precise weak formulation. The desired state \( u_d \) is in \( L^2(\Omega) \), see also further assumptions (\( u_d \in L^p(\Omega) \) or \( u_d \in L^\infty(\Omega) \)) below. The parameter \( \alpha \) is assumed to be positive.

This problem setting with the control from a measure space was considered in [10], where it has been observed that this setting leads to optimal controls with sparse structure. This is important for many applications, cf., e.g., [11]. For another functional analytic concept utilizing the \( L^1(\Omega) \)-norm of the control combined with a \( L^2 \)-regularization and/or with control constraints we refer e.g. to [18, 20, 9].

This paper is mainly concerned with the discretization of the problem (1.1) – (1.2). In [7] a discretization concept for this problem is presented and the following error estimates are derived:

\[
J(\bar{q}, \bar{u}) - J(\bar{q}_h, \bar{u}_h) = O(h^{2-\frac{d}{2}}) \quad \text{and} \quad \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} = O(h^{1-\frac{d}{2}}),
\]

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where \((\bar{q}, \bar{u})\) is the unique solution to (1.1) – (1.2), \(h\) is the discretization parameter and \((\bar{q}_h, \bar{u}_h)\) is the discrete solution. Our main contribution is the improvement of these estimates using the same discretization concept to

\[
J(\bar{q}, \bar{u}) - J(\bar{q}_h, \bar{u}_h) = \mathcal{O}(h^{4-d}|\ln h|^\gamma) \quad \text{and} \quad \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} = \mathcal{O}(h^{2-\frac{d}{2}}|\ln h|^\frac{\gamma}{2}), \quad (1.3)
\]

with \(\gamma = \frac{7}{2}\) for \(d = 2\) and \(\gamma = 1\) for \(d = 3\). Moreover we provide an estimate for the error in the control variable. Although one can only expect \(\bar{q}_h \rightharpoonup \bar{q}\) in \(\mathcal{M}(\Omega)\), see [7], we derive the following estimate with respect to the \(H^{-2}(\Omega)\)-norm:

\[
\|\bar{q} - \bar{q}_h\|_{H^{-2}(\Omega)} = \mathcal{O}(h^{2-\frac{d}{2}}|\ln h|^{\frac{\gamma}{2}}).
\]

We obtain these improved estimates with similar assumptions as in [7], but employing error estimates for the state solution in \(L^t(\Omega)\) for \(t < 2\), which are of (almost) optimal order, see Lemma 3.3, combined with a more careful study of the regularity of the state solutions for a measure valued right hand side. However, the assumption on the desired state \(u_d\) needs to be slightly stronger than in [7], see Remark 4.1 below.

The numerical examples (see Section 7) indicate that the estimates (1.3) are sharp. However, we make the following observation: In the two-dimensional case we see the predicted order of almost \(\mathcal{O}(h)\) with respect to the state variable in all examples. But for the three-dimensional case, the predicted order of (almost) \(\mathcal{O}(h^{\frac{2}{d}})\) is observed only in examples where the exact optimal controls contains Dirac measures. For optimal controls \(\bar{q}\) with better regularity, we observe convergence rates similar to the two-dimensional case. Motivated by this observation, we show in Section 2, that assuming a bounded desired state \(u_d \in L^\infty(\Omega)\) implies that \(\bar{u}\) must be bounded as well, which immediately rules out controls containing Dirac measures. Another direct consequence is \(\bar{q} \in H^{-1}(\Omega)\), which allows us to show an order of convergence of (almost) order \(\mathcal{O}(h^{\frac{2}{d}})\) for the state error \(\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}\) and \(d = 3\). Under the additional assumption that \(\bar{q} \in W^{-1,p}(\Omega)\) with \(p > 2\) this rate can be improved further, see Theorem 5.1.

The paper is structured as follows. In the next section we recall the optimality conditions from [10] and [7], discuss some consequences of them and prove that the optimal state \(\bar{u}\) is bounded provided that \(u_d \in L^\infty(\Omega)\). In Section 3 we describe the finite element discretization and derive some error estimates for the state equation. In Section 4 we prove the main estimates (1.3) and in Section 5 we derive an improved estimates for \(d = 3\) under an additional regularity assumption. In the last section we present numerical examples illustrating our results.

Throughout we will denote by \((\cdot, \cdot)\) the \(L^2(\Omega)\) inner product and by \((\cdot, \cdot)^\ast\) the duality product between \(\mathcal{M}(\Omega)\) and \(C_0(\Omega)\).

### 2. Optimality system and regularity.

As the first step we recall the weak formulation of the state equation (1.2). For a given \(q \in \mathcal{M}(\Omega)\) the solution \(u = u(q)\) is determined by

\[
u \in L^2(\Omega) : \langle u, -\Delta \varphi \rangle = \langle q, \varphi \rangle \quad \forall \varphi \in H^2(\Omega) \cap H^1_0(\Omega).
\]

It is well known, that the above formulation possesses a unique solution, which belongs to \(W^{1,s}_0(\Omega)\) for all \(1 \leq s < \frac{d}{d-1}\), see, e.g., [6]. Moreover, there holds the following stability estimate.

**Lemma 2.1.** For each \(0 < \varepsilon \leq \frac{1}{d-1}\) let \(s_\varepsilon\) be given as

\[
s_\varepsilon = \frac{d}{d-1} - \varepsilon.
\]
There exists a constant $c$ independent of $\varepsilon$, such that for all $q \in \mathcal{M}(\Omega)$ and the corresponding solution $u$ of (1.2) the following estimate holds:

$$\|u\|_{W^{1,s_{\varepsilon}}_0(\Omega)} \leq \frac{c}{\varepsilon} \|q\|_{\mathcal{M}(\Omega)}.$$

**Proof.** The estimate for $\|u\|_{W^{1,s_{\varepsilon}}_0(\Omega)}$ with an $s$-dependent constant is shown in [6]. To obtain the precise dependence of $\varepsilon$ we use the continuous embedding of $W^{1,s_{\varepsilon}}_0(\Omega)$ into $C_0(\Omega)$, there

$$\frac{1}{s^\prime_{\varepsilon}} + \frac{1}{s_{\varepsilon}} = 1, \quad s_{\varepsilon} > d.$$

From Theorem 8.10 (in the 3. edition) in [2] we obtain

$$\|v\|_{C_0(\Omega)} \leq \frac{c}{\varepsilon} \|v\|_{W^{1,s_{\varepsilon}}_0(\Omega)}$$

for all $v \in W^{1,s_{\varepsilon}}_0(\Omega)$ with the constant $c$ independent of $\varepsilon$. Using the result from [1], see also [15], we estimate

$$\|\nabla u\|_{L^{s_{\varepsilon}}(\Omega)} \leq c \sup_{v \in W^{1,s_{\varepsilon}}_0(\Omega)} \frac{\langle \nabla u, \nabla v \rangle}{\|\nabla v\|_{L^{s_{\varepsilon}}(\Omega)}} = c \sup_{v \in W^{1,s_{\varepsilon}}_0(\Omega)} \frac{\langle q, v \rangle}{\|\nabla v\|_{L^{s_{\varepsilon}}(\Omega)}} \leq \frac{c}{\varepsilon} \|q\|_{\mathcal{M}(\Omega)}.$$

This completes the proof. \(\square\)

Due to the embedding of $W^{1,s_{\varepsilon}}_0(\Omega)$ into $L^2(\Omega)$ for \(\frac{2d}{d+2} \leq s < \frac{d}{d-1}\) the cost functional (1.1) is well-defined. Moreover, the solution operator mapping $q \in \mathcal{M}(\Omega)$ to $u = u(q) \in L^2(\Omega)$ is injective and therefore the cost functional is strictly convex. Using this fact, the existence of a unique solution $(\bar{q}, \bar{u})$ to (1.1) - (1.2) can be directly obtained, see [10] for details.

The following optimality system is obtained in [10, 7].

**Theorem 2.2.** Let $(\bar{q}, \bar{u})$ be the solution to (1.1) - (1.2). Then there exists a unique adjoint state $\bar{z} \in H^2(\Omega) \cap H^1_0(\Omega) \rightarrow C_0(\Omega)$ satisfying

$$\begin{cases}
-\Delta \bar{z} = \bar{u} - u_d \text{ in } \Omega \\
\bar{z} = 0 \text{ on } \partial\Omega,
\end{cases} \quad (2.1)$$

and

$$-\langle q - \bar{q}, \bar{z} \rangle + \alpha \|q\|_{\mathcal{M}(\Omega)} \leq \alpha \|q\|_{\mathcal{M}(\Omega)} \text{ for all } q \in \mathcal{M}(\Omega). \quad (2.2)$$

Furthermore this implies

$$\|\bar{z}\|_{C_0(\Omega)} \leq \alpha, \quad (2.3)$$

the support of $\bar{q}$ is contained in the set $\{ x \in \Omega \mid |\bar{z}(x)| = \alpha \}$, and for the Jordan-decomposition $\bar{q} = \bar{q}^+ - \bar{q}^-$ we have

$$\text{supp } \bar{q}^+ \subset \{ x \in \Omega \mid \bar{z}(x) = -\alpha \} \quad \text{and} \quad \text{supp } \bar{q}^- \subset \{ x \in \Omega \mid \bar{z}(x) = \alpha \}. \quad (2.4)$$
Remark 2.3. The optimality condition (2.2) can be equivalently reformulated as

\begin{equation}
(u(q) - \bar{u}, \bar{u} - u_d) + \alpha (\|q\|_{\mathcal{M}(\Omega)} - \|\bar{q}\|_{\mathcal{M}(\Omega)}) \geq 0 \quad \text{for all } q \in \mathcal{M}(\Omega). \tag{2.5}
\end{equation}

The statement of the above theorem directly implies the following corollary on the structure of the optimal control \( \bar{q} \).

Corollary 2.4. There exist \( \gamma > 0 \) depending on the data of the problem, such that

\begin{equation}
\text{supp } \bar{q} \subset \Omega_\gamma = \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) > \gamma \}, \tag{2.6}
\end{equation}

and additionally

\begin{equation}
\text{dist}(\text{supp } \bar{q}^+, \text{supp } \bar{q}^-) > \gamma. \tag{2.7}
\end{equation}

The first property implies that the support is compact.

Proof. The adjoint state \( \bar{z} \) belongs to \( H^2(\Omega) \hookrightarrow C_{0,\beta}^0(\bar{\Omega}) \) with some \( \beta > 0 \). This implies (due to the homogeneous Dirichlet boundary conditions) the existence of \( \gamma > 0 \), such that

\[ |\bar{z}(x)| \leq \frac{\alpha}{2} \quad \text{on } \Omega \setminus \Omega_\gamma. \]

We complete the first part of the proof using the statement on the support of \( \bar{q} \) from Theorem 2.2. With a similar argument we derive the second statement, since due to (2.4), the adjoint state attains the values \( \pm \alpha \) respectively on the support of \( \bar{q}^- \) and \( \bar{q}^+ \).

Finally, we will derive an additional regularity for \( \bar{u} \) if the desired state \( u_d \) is bounded.

Theorem 2.5. Assume that the desired state \( u_d \) is in \( L^\infty(\Omega) \). Then the optimal state \( \bar{u} \) is also in \( L^\infty(\Omega) \) and there holds

\begin{equation}
\|\bar{u}\|_{L^\infty(\Omega)} \leq \|u_d\|_{L^\infty(\Omega)}. \nonumber
\end{equation}

A direct consequence of this theorem is an additional regularity for the optimal control \( \bar{q} \) and for the optimal state \( \bar{u} \).

Corollary 2.6. Assume that the desired state \( u_d \) is in \( L^\infty(\Omega) \). Then the optimal state \( \bar{u} \) lies in \( H^1_0(\Omega) \cap L^\infty(\Omega) \) and the optimal control \( \bar{q} \) lies in \( H^{-1}(\Omega) \). There holds

\begin{equation}
\|\nabla \bar{u}\|_{L^2(\Omega)}^2 \leq \|q\|_{\mathcal{M}(\Omega)} \|u_d\|_{L^\infty(\Omega)} \quad \text{and} \quad \|q\|_{H^{-1}(\Omega)} = \|\nabla \bar{u}\|_{L^2(\Omega)}. \tag{2.8}
\end{equation}

In order to prove Theorem 2.5 and Corollary 2.6 we use some results from potential theory: First, introduce the Green’s function \( G_\Omega : \Omega \times \Omega \to \mathbb{R}^+ \cup \{+\infty\} \) as in e.g. [3] or [14]. Then, for a positive measure \( \mu \in \mathcal{M}(\Omega) \), \( \mu \geq 0 \) we define the numeric function \( v^* : \Omega \to \mathbb{R}^+ \cup \{+\infty\} \) by

\[ v^* = S(\mu) := \int_{\Omega} G_\Omega(\cdot, y) \, d\mu(y), \tag{2.8} \]

which is subharmonic and thus lower semicontinuous (see again [3]). If we normalize \( G_\Omega \) by the right constant, we obtain the following simple result.
Lemma 2.7. For a compactly supported \( \mu \in \mathcal{M}(\Omega) \), \( \mu \geq 0 \) the weak solution \( v \in W^{1,s}_0(\Omega) \) with \( 1 \leq s < \frac{d}{d-1} \) to the problem
\[
-\Delta v = \mu \text{ in } \Omega, \\
v = 0 \text{ on } \partial \Omega,
\] is equal to \( v^* = S(\mu) \) (Lebesgue-)almost everywhere.

Proof. With [3, Theorem 4.3.8] the function \( v^* \) is a distributional solution of (2.9), and by a density argument, it is also a weak solution, unique in an almost everywhere sense. \( \square \)

With the help of the above lemma, we obtain a pointwise representative of the optimal solution \( u^* : \Omega \to \mathbb{R} \cup \{-\infty, \infty\} \), defined as
\[
u^* := S(\bar{\mu}^+) - S(\bar{\mu}^-) = S(\bar{\mu}).
\]

Due to (2.6) the measures \( \bar{\mu}^+ \) and \( \bar{\mu}^- \) are compactly supported, and with (2.7) \( u^* \) is well defined with values in \( \mathbb{R} \cup \{-\infty, \infty\} \). With Lemma 2.7 we easily derive that \( u^* = \bar{u} \) almost everywhere.

The next lemma states (roughly speaking), that if the optimal state is bounded on \( \text{supp } \bar{\mu} \), then it is bounded everywhere on \( \Omega \) by the same constant. For positive measures in \( \mathcal{M}(\Omega) \) this statement can be directly obtained from [14, Theorem 1.6'] in the two-dimensional case. For \( d = 3 \), the analogous theorem, see [14, Theorem 1.10], is stated only for \( \Omega = \mathbb{R}^d \). Therefore, we provide a direct proof.

Lemma 2.8. Let \( \bar{\mu} \in \mathcal{M}(\Omega) \) be the optimal control. If \( u^* = S(\bar{\mu}) \) is bounded from above by some constant \( C^+ \geq 0 \) on \( \text{supp } \bar{\mu}^+ \), then it is bounded everywhere by \( C^+ \). Analogously, if \( u^* \) is bounded from below by some \( C^- \leq 0 \) on \( \text{supp } \bar{\mu}^- \), then \( u^* \) is bounded from below everywhere by \( C^- \).

Proof. Suppose \( u^* \leq C^+ \) on \( \text{supp } \bar{\mu}^+ \). With (2.7) we estimate
\[
S(\bar{\mu}^+) = u^* + S(\bar{\mu}^-) \leq C^+ + c_\gamma \|\bar{\mu}^-\|_{\mathcal{M}(\Omega)} \text{ on } \text{supp } \bar{\mu}^+,
\]
where \( c_\gamma = c \log(\frac{1}{\gamma} \text{diam } \Omega) \) for \( d = 2 \) and \( c_\gamma = \frac{c_\gamma}{\gamma} \) for \( d = 3 \) due to the growth properties of the Green’s function. Thus, \( S(\bar{\mu}^+) \) is bounded on \( \text{supp } \bar{\mu}^+ \) as well. With [3, Corollary 4.5.2] we can now construct a sequence of compact sets \( \{K_i\} \) with
\[
\bar{\mu}^+(\text{supp } \bar{\mu}^+ \setminus K_i) \to 0 \text{ for } i \to \infty,
\]
such that the functions \( S(\bar{\mu}^+|_{K_i}) \) are continuous.

Now, we consider the solutions
\[
u_i = S(\bar{\mu}^+|_{K_i}) - S(\bar{\mu}^-) \leq u^*.
\]
Recalling that \( -S(\bar{\mu}^-) \) is upper semicontinuous, we obtain that each \( u_i \) is upper semicontinuous as well. For each \( x_0 \) on the boundary of \( \Omega \setminus \text{supp } \bar{\mu}^+ \), which is a subset of \( \text{supp } \bar{\mu}^+ \cup \partial \Omega \), we have \( u_i(x_0) \leq u^*(x_0) \leq C^+ \) and with upper semicontinuity
\[
\limsup_{x \to x_0} u_i(x) \leq C^+.
\]
Using the fact that \( u_i \) is subharmonic on \( \Omega \setminus \text{supp } \mu^+ \) and the condition (2.11) we apply the maximum principle for subharmonic functions [3, Theorem 3.1.5], and obtain that \( u_i \) is bounded by \( C^+ \) everywhere on \( \Omega \) for every \( i \).
To complete the proof, it remains to show the convergence $u_i(x) \to u^*(x)$ for all $x \in \Omega \setminus \operatorname{supp} \bar{q}^+$. Let $x \in \Omega \setminus \operatorname{supp} \bar{q}^+$ be fixed. We denote by $\delta = \operatorname{dist}(x, \operatorname{supp} \bar{q}^+) > 0$. There holds

$$|u_i(x) - u^*(x)| = |S(\bar{q}^+|_{K_i})(x) - S(\bar{q}^+)(x)| \leq c_0 \bar{q}^+(\operatorname{supp} \bar{q}^+ \setminus K_i) \to 0, \quad i \to \infty,$$

where we have again used growth properties of the Green’s function and (2.10).

The second statement is proved completely analogously. \( \Box \)

With these preparations we can give proofs of the claimed results.

**Proof.** [Proof of Corollary 2.6] Assume the contrary, i.e., that we have $C, \varepsilon > 0$, such that $|u_d| \leq C$ almost everywhere in $\Omega$, but $|\bar{u}| \geq C + \varepsilon$ on some set of positive Lebesgue measure. Without loss of generality, we can assume that

$$|\{ x \in \Omega \mid \bar{u}(x) \geq C + \varepsilon \}| > 0.$$

Let $u^* = S(\bar{q})$, which necessarily must be larger than $C + \varepsilon$ for some $x \in \operatorname{supp} \bar{q}^+$ with Lemma 2.8. In the ball $B_\gamma(x)$ we have with Corollary 2.4 that $\bar{q}^+|_{B_\gamma(x)} = 0$ and therefore $S(\bar{q}|_{B_\gamma(x)})$ is lower semicontinuous. We decompose

$$u^* = S(\bar{q}|_{B_\gamma(x)}) + S(\bar{q}|_{\Omega \setminus B_\gamma(x)})$$

and obtain that $S(\bar{q}|_{\Omega \setminus B_\gamma(x)})$ is harmonic and consequently continuous on $B_\gamma(x)$. This implies the lower semicontinuity of $u^*$ on $B_\gamma(x)$. This means, the set

$$\{ y \in B_\gamma(x) \mid u^*(y) > C + \varepsilon \}$$

is open, and we can find a radius $r > 0$ such that $\bar{u} \geq C + \varepsilon$ almost everywhere in the ball $B_r(x)$.

Note that $x \in \operatorname{supp} \bar{q}^+$ implies $\bar{z}(x) = -\alpha$ with Theorem 2.2. We define $w$ to be the solution to

$$-\Delta w = \varepsilon \quad \text{in } B_r(x),$$

$$w = 0 \quad \text{on } \partial B_r(x),$$

which is clearly strictly positive at $x$. Considering the minimum principle for $\bar{z} = \bar{z} - w$ which solves

$$-\Delta \bar{z} = \bar{u} - u_d - \varepsilon \geq 0 \quad \text{in } B_r(x),$$

$$\bar{z} = \bar{z} \quad \text{on } \partial B_r(x),$$

we see that the minimum value $z_{\min} = \inf_{x \in B_r(x)} \bar{z}(x)$ must be attained for some $x' \in \partial B_r(x)$. Comparing with the center $x$ we find

$$\bar{z}(x') = \bar{z}(x') = (\bar{z} - w)(x') \leq (\bar{z} - w)(x) < \bar{z}(x) = -\alpha,$$

which is a violation of the bounds on the adjoint state (2.3) and thus a contradiction. \( \Box \)

**Proof.** [Proof of Corollary 2.6] The result can be derived by considering a sequence of smooth approximations to $\bar{q}$, testing the corresponding state equation with the smooth solution and a subsequential weak limit argument.

However, the statement directly follows from a well-known classical result: Since, by the previous theorem, $u^*$ is bounded, we can pair $u^*$ with $\bar{q}$ to obtain

$$\|\bar{q}\|_{\mathcal{M}(\Omega)}\|u^*\|_{L^\infty(\Omega)} \geq \langle \bar{q}, u^* \rangle = \int_\Omega u^*(x) \, d\bar{q}(x) = \int_\Omega \int_\Omega G_{\Gamma}(x,y) \, d\bar{q}(x) \, d\bar{q}(y).$$
With [14, Theorem 1.20], this implies $\nabla u^* \in L^2(\Omega)$ and
\[
\int_{\Omega} \int_{\Omega} G_{\Omega}(x, y) \, dq(x) \, dq(y) = \|\nabla u^*\|_{L^2(\Omega)}^2,
\]
which implies the first part of the claim. The second assertion is evident.

3. Discretization. For the discretization of the state equation we use linear finite elements on a family of shape regular quasi-uniform triangulations $\{T_h\}_h$, see, e.g., [4]. The discretization parameter $h$ denotes the maximal diameter of cells $K \in T_h$.

We set
\[
\bar{\Omega}_h = \bigcup_{K \in T_h} \bar{K}
\]
and make the usual assumption
\[
|\Omega \setminus \Omega_h| \leq ch^2.
\]
The finite element space associated with $T_h$ is defined as usual by
\[
V_h = \{ v_h \in C_0(\Omega) \mid v_h|_K \in P_1(K) \text{ for all } K \in T_h \text{ and } v_h = 0 \text{ on } \Omega \setminus \Omega_h \}.
\]
For a given $q \in M(\Omega)$ the discrete solution $u_h = u_h(q)$ is determined by
\[
u h \in V_h : (\nabla u_h, \nabla v_h) = \langle q, v_h \rangle \text{ for all } v_h \in V_h. \tag{3.1}
\]
To define the approximation of the optimal control problem (1.1) – (1.2) we follow the approach from [7] and do not discretize the control space, cf. the variational approach by [13]. The discrete optimal control problem is then given as
\[
\text{Minimize } J(q_h, u_h), \quad q_h \in M(\Omega) \text{ and subject to (3.1)}. \tag{3.2}
\]
The existence of a solution can be shown as on the continuous level. The optimal state $\bar{u}_h$ is unique. The discrete solution operator mapping $q \in M(\Omega)$ to $u_h(q)$ is not injective and the uniqueness of the optimal control can not be guaranteed. However, one special solution can be identified, which is numerically accessible, see [7] and the discussion below.

By $\{x_i\}, i = 1, 2, \ldots, N_h$ we denote the interior nodes of $\Omega_h$ and by $\{e_i\} \subset V_h$ the corresponding node basis functions. We introduce the space $M_h$ consisting of linear combination of Dirac functionals associated with the nodes $x_i$:
\[
M_h = \left\{ q_h \in M(\Omega) \mid q_h = \sum_{i=1}^{N_h} \beta_i \delta_{x_i}, \beta_i \in \mathbb{R}, i = 1, 2, \ldots, N_h \right\}
\]
and an operator $\Lambda_h : M(\Omega) \rightarrow M_h$ (see [7]) by
\[
\Lambda_h q = \sum_{i=1}^{N_h} (q, e_i) \delta_{x_i}.
\]
There holds the following theorem, see [7].
Theorem 3.1. Among the solutions to (3.2) there exists a unique solution \( \bar{q}_h \in \mathcal{M}_h \) with the corresponding state \( \bar{u}_h = u_h(\bar{q}_h) \). Any other solution \( \hat{q}_h \in \mathcal{M}(\Omega) \) satisfies \( \Lambda_h \hat{q}_h = \bar{q}_h \). Moreover there holds
\[
\bar{q}_h \xrightarrow{h} \bar{q} \text{ in } \mathcal{M}(\Omega) \quad \text{and} \quad \|\bar{q}_h\|_{\mathcal{M}(\Omega)} \to \|\bar{q}\|_{\mathcal{M}(\Omega)}
\]
for \( h \to 0 \).

For the solution \( (\bar{q}_h, \bar{u}_h) \) from this theorem the following discrete version of the optimality conditions holds, which can be derived as in the continuous case, cf. [7].

Theorem 3.2. Let \( (\bar{q}_h, \bar{u}_h) \in \mathcal{M}_h \times V_h \) be the discrete solution, see Theorem 3.1. Then there exists the discrete adjoint state \( \bar{z}_h \in V_h \) fulfilling
\[
(\nabla v_h, \nabla \bar{z}_h) = (\bar{u}_h - u_d, v_h) \quad \text{for all } v_h \in V_h
\]
and the optimality condition
\[
-(q - \bar{q}_h, \bar{z}_h) + \alpha \|\bar{q}_h\|_{\mathcal{M}(\Omega)} \leq \alpha \|q\|_{\mathcal{M}(\Omega)} \quad \text{for all } q \in \mathcal{M}(\Omega). \tag{3.3}
\]
The last condition can be equivalently rewritten as
\[
(u_h(q) - \bar{u}_h, \bar{u}_h - u_d) + \alpha \left( \|q\|_{\mathcal{M}(\Omega)} - \|\bar{q}_h\|_{\mathcal{M}(\Omega)} \right) \geq 0 \quad \text{for all } q \in \mathcal{M}(\Omega), \tag{3.4}
\]
cf. Remark 2.3.

In order to prove our main result mentioned in the introduction, we first provide some estimates for the error \( u(q) - u_h(q) \) for a fixed control \( q \in \mathcal{M}(\Omega) \).

Lemma 3.3. Let \( q \in \mathcal{M}(\Omega) \) with associated continuous and discrete states \( u = u(q) \) and \( u_h = u_h(q) \) be given. Then there holds:

(i) \( \|u - u_h\|_{L^p(\Omega)} \leq c_p h^{2+\frac{d}{p}} \|q\|_{\mathcal{M}(\Omega)}, \quad p \in (1, \infty), \quad \frac{1}{p} + \frac{1}{p'} = 1 \)

(ii) \( \|u - u_h\|_{L^{p'}(\Omega)} \leq ch^2 |\ln h|^r \|q\|_{\mathcal{M}(\Omega)} \)

with \( r = 2 \) for \( d = 2 \) and \( r = \frac{1}{4} \) for \( d = 3 \).

Proof.

(i): For the first estimate in case \( p = 2 \) we refer, e.g., to [5]. For a general case, \( p \in (1, \infty) \) we set \( c = u - u_h \) and
\[
g_p(x) = |e(x)|^{p-1} \text{sgn}(e(x)).
\]
By a direct calculation it follows \( g_p \in L^{p'}(\Omega) \) and
\[
\|g_p\|_{L^{p'}(\Omega)} = \|e\|_{L^{p'}(\Omega)}^{p-1}.
\]
We consider a dual problem
\[
w \in H_0^1(\Omega) : (\nabla w, \nabla v) = (g_p, v) \quad \text{for all } v \in H_0^1(\Omega)
\]
and its Ritz projection
\[
w_h \in V_h : (\nabla w_h, \nabla v_h) = (g_p, v_h) \quad \text{for all } v_h \in V_h.
\]
By the elliptic regularity we obtain \( w \in W^{2,p'}(\Omega) \) and the corresponding \( L^\infty \)-estimate gives
\[
\|w - w_h\|_{C_0(\Omega)} \leq c_p h^{2-\frac{d}{p'}} \|
abla^2 w\|_{L^{p'}(\Omega)} \leq c_p h^{2-\frac{d}{p'}} \|g_p\|_{L^{p'}(\Omega)} \leq c_p h^{2-\frac{d}{p'}} \|e\|_{L^p(\Omega)}^{p-1}.
\]
For the error $\|e\|_{L^p(\Omega)}$ we obtain

$$
\|e\|_{L^p(\Omega)}^p = (e, g_p) = (\nabla e, \nabla w) \\
= (\nabla e, \nabla (w - w_h)) = (\nabla u, \nabla (w - w_h)) \\
= (q, w - w_h) \leq \|q\|_{M(\Omega)} \|w - w_h\|_{C_0(\Omega)} \\
\leq c_p h^{2 - \frac{2}{p}} \|q\|_{M(\Omega)} \|e\|_{L^p(\Omega)}^{p-1},
$$

which gives the desired estimate.

(ii): To obtain the second estimate, we set $g_1 = \text{sgn}(e) \in L^\infty(\Omega)$. There holds

$$
\|e\|_{L^1(\Omega)} = (e, g_1) = (\nabla e, \nabla w) \\
= (\nabla e, \nabla (w - w_h)) = (\nabla u, \nabla (w - w_h)) \\
\leq \|q\|_{M(\Omega)} \|w - w_h\|_{C_0(\Omega)}.
$$

For the pointwise error in $w$ we use the result from Rannacher and Frehse [12] for $d = 2$ and Rannacher [16] for $d = 3$ and obtain:

$$
\|w - w_h\|_{C_0(\Omega)} \leq c h^2 |\ln h|^r \|g_1\|_{L^\infty(\Omega)}.
$$

This completes the proof.

Assuming higher regularity for $\bar{q}$, we can also give the following estimate, which will be needed later on in Section 5 for the improved error estimates.

**Lemma 3.4.** Let $q \in W^{-1,p}(\Omega)$ for $1 < p < \infty$. Then we have with $u = u(q)$ and $u_h = u_h(q)$ as before that

$$
\|u - u_h\|_{L^p(\Omega)} \leq c h \|q\|_{W^{-1,p}(\Omega)}.
$$

**Proof.** Due to the assumption we have $u \in W^{1,p}(\Omega)$ and $\|\nabla u\|_{L^p(\Omega)} \leq c \|q\|_{W^{-1,p}(\Omega)}$ with the result from [1], cf. the proof of Lemma 2.1. Defining $e = u - u_h$ and $g_p$, $w$, $w_h$ as in the proof of Lemma 3.3 (i) we obtain

$$
\|e\|_{L^p(\Omega)}^p = (\nabla u, \nabla (w - w_h)) \\
\leq \|\nabla u\|_{L^p(\Omega)} \|\nabla (w - w_h)\|_{L^p(\Omega)} \\
\leq c h \|q\|_{W^{-1,p}(\Omega)} \|g_p\|_{L^p(\Omega)} = c h \|q\|_{W^{-1,p}(\Omega)} \|e\|_{L^p(\Omega)}^{p-1}
$$

with $\frac{1}{p} + \frac{1}{p'} = 1$ and a standard error estimate, which yields the desired result. □
Another useful result concerns the growth behavior of discrete solutions in the limiting cases of the Sobolev embedding theorem

\[ \| u_h \|_{L^t(\Omega)} \leq c_t \| q \|_{\mathcal{M}(\Omega)} \quad \text{for all } t < \frac{d}{d-2}. \]

For the discrete solutions, we have the following result.

**Lemma 3.5.** Let \( q \in \mathcal{M}(\Omega) \) with the discrete solution \( u_h = u_h(q) \) as above. Then we have

\[ \| u_h \|_{L^\infty(\Omega)} \leq c |\ln h|^{\frac{3}{2}} \| q \|_{\mathcal{M}(\Omega)} \quad \text{for } d = 2, \]

\[ \| u_h \|_{L^3(\Omega)} \leq c |\ln h| \| q \|_{\mathcal{M}(\Omega)} \quad \text{for } d = 3. \]

**Proof.** In the first step we estimate

\[ \| u_h \|_{L^\infty(\Omega)} \leq c |\ln h|^{\frac{1}{2}} \| \nabla u_h \|_{L^2(\Omega)} \quad \text{for } d = 2, \]

by the discrete Sobolev inequality, see [4], and

\[ \| u_h \|_{L^d(\Omega)} \leq c \| \nabla u_h \|_{L^{\frac{d}{d-1}}(\Omega)} \quad \text{for } d = 3, \]

by the Sobolev embedding. Defining \( \sigma = \frac{d}{d-1} \) (\( \sigma = 2 \) and \( \sigma = \frac{3}{2} \) for 2d and 3d respectively), we proceed in a common way with an inverse estimate and the stability of the Ritz projection with respect to the \( W^{1,s} \)-seminorm, see [4],

\[ \| \nabla u_h \|_{L^s(\Omega)} \leq c h^{\frac{d}{2} - \frac{d}{s}} \| \nabla u_h \|_{L^{s}(\Omega)} \]

\[ \quad \leq c h^{\frac{d}{2} - \frac{d}{s}} \| \nabla u \|_{L^{s}(\Omega)}, \]

for any \( 1 < s < \sigma \), where the constant \( c \) is independent of \( s \). Then we choose \( s = s_\varepsilon = \sigma - \varepsilon \) for \( 0 < \varepsilon < \sigma - 1 \), which implies that

\[ \frac{d}{\sigma} - \frac{d}{s_\varepsilon} = - \frac{d\varepsilon}{\sigma(\sigma - \varepsilon)} > -\varepsilon \sigma^{-1} = -\varepsilon(d-1). \]

We obtain by Lemma 2.1

\[ \| \nabla u_h \|_{L^s(\Omega)} \leq c \varepsilon h^{-\varepsilon(d-1)} \| q \|_{\mathcal{M}(\Omega)}. \]

Choosing now \( \varepsilon = 1/|\ln h| \) we obtain

\[ \| \nabla u_h \|_{L^\infty(\Omega)} \leq c |\ln h| \| q \|_{\mathcal{M}(\Omega)}, \]

which, together with the first estimate, completes the proof. \( \square \)

**4. General error estimates.** In the next theorem we provide an error estimate for the error with respect to the cost functional. To state this theorem we need an assumption on the desired state \( u_d \).

**Assumption 1.** We assume

\[ u_d \in \begin{cases} L^\infty(\Omega), & \text{for } d = 2 \\ L^3(\Omega), & \text{for } d = 3. \end{cases} \]
Lemma 3.3: The other terms are estimated separately in 2d and in 3d. For the second term in (4.1) we obtain by the estimate (i) assumption in [7], where terms in (4.1) to obtain and then to apply this estimate for both

\[ J(\bar{q}, \bar{u}) \leq J(q, u(q)) \quad \text{and} \quad J(\bar{q}, \bar{u}_h) \leq J(q, u_h(q)). \]

Consequently we have

\[ J(q, u(q)) - J(q, u_h(q)) \]

Therefore, it remains to estimate the error with respect to the cost functional for a fixed \( q \in \mathcal{M}(\Omega) \), i.e.

\[ |J(q, u(q)) - J(q, u_h(q))| \leq \frac{1}{2} \|q - u\|_{L^2(\Omega)} - \frac{1}{2} \|u_h - u_d\|_{L^2(\Omega)} \]

and then to apply this estimate for both \( q = \bar{q} \) and \( q = \bar{q}_h \).

For fixed \( q \in \mathcal{M}(\Omega) \) we now use the notation \( u = u(q) \) and \( u_h = u_h(q) \). There holds:

\[ J(q, u) - J(q, u_h) = \frac{1}{2} \|u - u_h\|_{L^2(\Omega)} - \frac{1}{2} \|u_h - u_d\|_{L^2(\Omega)} \]

\[ = \frac{1}{2} (u - u_h, u + u_h - 2u_d) \quad \text{(4.1)} \]

\[ = -(u - u_h, u_d) + \frac{1}{2} \|u - u_h\|_{L^2(\Omega)}^2 + (u - u_h, u_h). \]

For the second term in (4.1) we obtain by the estimate (i) for \( p = 2 \) from Lemma 3.3

\[ \|u - u_h\|_{L^2(\Omega)}^2 \leq ch^{4-d} \|q\|^2_{\mathcal{M}(\Omega)}. \]

The other terms are estimated separately in 2d and in 3d.

The case \( d = 2 \). The first and last terms in (4.1) are estimated using (ii) from Lemma 3.3:

\[ (u - u_h, u_d) \leq \|u - u_h\|_{L^4(\Omega)} \|u_d\|_{L^\infty(\Omega)} \leq ch^2 \|q\|^2_{\mathcal{M}(\Omega)}, \]

\[ (u - u_h, u_h) \leq \|u - u_h\|_{L^4(\Omega)} \|u_h\|_{L^\infty(\Omega)} \leq ch^2 \|q\|^2_{\mathcal{M}(\Omega)} \|u_h\|_{L^\infty(\Omega)}. \]

Additionally, by Lemma 3.5 we have \( \|u_h\|_{L^\infty(\Omega)} \leq \ln h \|q\|^2_{\mathcal{M}(\Omega)}. \)

The case \( d = 3 \). Now, we use (i) for \( p = \frac{3}{2} \) from Lemma 3.3 for the remaining terms in (4.1) to obtain

\[ (u - u_h, u_d) \leq \|u - u_h\|_{L^4(\Omega)} \|u_d\|_{L^\infty(\Omega)} \leq ch \|q\|_{\mathcal{M}(\Omega)}, \]

\[ (u - u_h, u_h) \leq \|u - u_h\|_{L^4(\Omega)} \|u_h\|_{L^\infty(\Omega)} \leq ch \|q\|_{\mathcal{M}(\Omega)} \|u_h\|_{L^\infty(\Omega)}. \]
We apply again Lemma 3.5 and complete the proof. □

Remark 4.3. Assumption I excludes the case, where the desired state $u_d$ is given as a Green’s function. However, for construction of irregular examples with known exact solutions (see Section 7), it is desirable to choose $u_d$ to be the solution of

$$-Δu_d = δ_{x_0} \text{ in } Ω,$$

$$u_d = 0 \text{ on } ∂Ω,$$

with some $x_0 ∈ Ω$. For this choice of $u_d$ there holds:

$$u_d ∈ L^p(Ω) \text{ for all } p ∈ (1, ∞) \text{ for } d = 2$$

and

$$u_d ∈ L^{1−ε}(Ω) \text{ for all } ε ∈ (0, 1) \text{ for } d = 3.$$

The result of Theorem 4.2 can be directly extended to this situation. In this case an additional logarithmic term $|ln h|$ will appear.

In the next theorem we prove the main estimate for the error in the state variable, as announced in (1.3).

Theorem 4.4. Let the conditions of Theorem 4.2 be fulfilled. Then there holds

$$∥\bar{u} − \bar{u}_h∥_{L^2(Ω)} ≤ ch^{2−\frac{2}{d}}|ln h|^\frac{2}{d}.$$

Proof. We use the optimality condition (2.5), choose $q = \bar{q}_h$ and obtain

$$(u(\bar{q}_h) − \bar{u}, \bar{u} − u_d) + \alpha (∥\bar{q}_h∥_{M(Ω)} − ∥\bar{q}∥_{M(Ω)}) ≥ 0.$$

For the corresponding discrete optimality condition (3.4) we choose $q = \bar{q}$ resulting in

$$(u_h(\bar{q}) − \bar{u}_h, \bar{u}_h − u_d) + \alpha (∥\bar{q}∥_{M(Ω)} − ∥\bar{q}_h∥_{M(Ω)}) ≥ 0.$$

Adding these two inequalities we arrive at

$$(u(\bar{q}_h) − \bar{u}, \bar{u} − u_d) + (u_h(\bar{q}) − \bar{u}_h, \bar{u}_h − u_d) ≥ 0.$$

Rearranging the terms we obtain

$$(\bar{u}_h − \bar{u}, \bar{u} − u_d) + (u(\bar{q}_h) − \bar{u}_h, \bar{u} − u_d) + (\bar{u} − \bar{u}_h, \bar{u}_h − u_d) + (u_h(\bar{q}) − \bar{u}, \bar{u}_h − u_d) ≥ 0$$

resulting in

$$∥\bar{u} − \bar{u}_h∥_{L^2(Ω)}^2 ≤ (u(\bar{q}_h) − \bar{u}_h, \bar{u} − u_d) + (u_h(\bar{q}) − \bar{u}, \bar{u}_h − u_d)$$

$$= (u(\bar{q}_h) − \bar{u}_h, \bar{u} − u_d) + (u_h(\bar{q}) − \bar{u}_h, \bar{u}_h − u_d) + (u_h(\bar{q}) − \bar{u}, \bar{u}_h − u_d). \quad (4.2)$$

For the first term in (4.2) we obtain by the estimate (i) for $p = 2$ from Lemma 3.3

$$(u(\bar{q}_h) − \bar{u}_h, \bar{u} − u_d(\bar{q})) ≤ ∥u(\bar{q}_h) − \bar{u}_h∥_{L^2(Ω)} ∥\bar{u} − u_d(\bar{q})∥_{L^2(Ω)} ≤ ch^{2−d}∥\bar{q}∥_{M(Ω)} ∥\bar{q}_h∥_{M(Ω)}.$$

The second and the third terms in (4.2) are estimated by the same procedure as in the proof of Theorem 4.2 resulting in

$$∥\bar{u} − \bar{u}_h∥_{L^2(Ω)}^2 ≤ ch^{2−d}|ln h|^\gamma.$$
This completes the proof. ☐

With help of this result, we can also provide an estimate for the error of the control in $H^{-2}(\Omega)$.

**Theorem 4.5.** Let the conditions of Theorem 4.2 be fulfilled. Then there holds

$$
\|\bar{q} - \tilde{q}_h\|_{H^{-2}(\Omega)} \leq ch^{2-\frac{d}{2}}|\ln h|^\frac{1}{2}.
$$

**Proof.** For a given $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ and the nodal interpolation $i_h \psi \in V_h$ we have $(\bar{q}_h, \psi) = (\tilde{q}_h, i_h \psi)$ since $\tilde{q}_h$ is a linear combination of nodal Dirac delta functions and we obtain

$$
\langle \bar{q} - \bar{q}_h, \psi \rangle = \langle \bar{q} - \tilde{q}_h, i_h \psi \rangle + \langle \tilde{q}_h, \psi - i_h \psi \rangle
= \langle \nabla (\bar{u} - \bar{u}_h), \nabla i_h \psi \rangle + \langle \bar{q}, \psi - i_h \psi \rangle
= \langle \nabla (\bar{u} - \bar{u}_h), \nabla \psi \rangle - \langle \nabla (\bar{u} - \bar{u}_h), \nabla (\psi - i_h \psi) \rangle + \langle \bar{q}, \psi - i_h \psi \rangle.
$$

For the first term we get by Theorem 4.4

$$
\langle \nabla (\bar{u} - \bar{u}_h), \nabla \psi \rangle = (\bar{u} - \bar{u}_h, -\Delta \psi) \leq ch^{2-\frac{d}{2}}|\ln h|^\frac{2}{2} \|\psi\|_{H^2(\Omega)}.
$$

For the second term we obtain

$$
\langle \nabla (\bar{u} - \bar{u}_h), \nabla (\psi - i_h \psi) \rangle \leq c(\|\nabla \bar{u}\|_{L^s(\Omega)} + \|\nabla \bar{u}_h\|_{L^s(\Omega)}) \|\nabla (\psi - i_h \psi)\|_{L^{s'}(\Omega)}
$$

for all $s'$ with $\frac{1}{s'} + \frac{1}{s} = 1$ and $s < \frac{d}{d-1}$. Using the interpolation estimate

$$
\|\nabla (\psi - i_h \psi)\|_{L^{s'}(\Omega)} \leq ch^{1-\frac{d}{2}+\frac{d}{s'}}\|\psi\|_{H^2(\Omega)},
$$

choosing $s = s_\varepsilon = \frac{d}{d-1} - \varepsilon$ for $0 < \varepsilon < \frac{1}{2}$ and exploiting the estimate from Lemma 2.1 we obtain

$$
\langle \nabla (\bar{u} - \bar{u}_h), \nabla (\psi - i_h \psi) \rangle \leq \frac{c}{\varepsilon} h^{2-\frac{d}{2}+4\varepsilon}(\|\bar{q}\|_{M(\Omega)} + \|\tilde{q}_h\|_{M(\Omega)}) \|\psi\|_{H^2(\Omega)}.
$$

The choice $\varepsilon = \frac{1}{|\ln h|}$ yields:

$$
\langle \nabla (\bar{u} - \bar{u}_h), \nabla (\psi - i_h \psi) \rangle \leq ch^{2-\frac{d}{2}}|\ln h| \|\psi\|_{H^2(\Omega)}.
$$

For the third term we get

$$
\langle \bar{q}, \psi - i_h \psi \rangle \leq c\|\bar{q}\|_{M(\Omega)} \|\psi - i_h \psi\|_{C^0(\Omega)} \leq ch^{2-\frac{d}{2}}|\ln h| \|\psi\|_{H^2(\Omega)}.
$$

This completes the proof. ☐

**5. Improved error estimates.** In the following we exploit the additional regularity derived in Section 2 to provide an improved estimate under the assumption that $u_d$ is bounded.

**Theorem 5.1.** In the case $d = 3$, let $(\bar{q}, \bar{u})$ be the solution to (1.1) - (1.2) and $(\tilde{q}_h, \tilde{u}_h) \in M_h \times V_h$ be the discrete solution, see Theorem 3.1. Let moreover $u_d \in L^\infty(\Omega)$, which implies $\bar{u} \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $\bar{q} \in H^{-1}(\Omega)$ with Theorems 2.5 and 2.6. Then there holds

$$
\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \leq ch^\frac{d}{2}|\ln h|^\frac{1}{2}.
$$
Furthermore, under the additional assumption \( \bar{q} \in W^{-1,p}(\Omega) \) for some \( p > 2 \), we obtain

\[
\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \leq c h^{\frac{1}{2}(1+\theta)}|\ln h|^{\frac{1}{2}+\theta}, \quad \text{where} \quad \theta = \frac{2 - \frac{1}{p}}{1 - \frac{1}{p}}.
\]

**Proof.** First, we obtain an \( L^2(\Omega) \) estimate for \( \bar{u} \) in terms of an \( L^\infty(\Omega) \) estimate for \( \bar{z} \). For that, we use the optimality condition (2.2), choosing \( q = \bar{q}_h \)

\[-(\bar{q}_h - \bar{q}, \bar{z} - \bar{z}_h) + \alpha \|\bar{q}_h\|_{\mathcal{M}(\Omega)} \leq \alpha \|\bar{q}_h\|_{\mathcal{M}(\Omega)},
\]

and the optimality condition (3.3) choosing \( q = \bar{q} \)

\[-(\bar{q} - \bar{q}_h, \bar{z} - \bar{z}_h) + \alpha \|\bar{q}_h\|_{\mathcal{M}(\Omega)} \leq \alpha \|\bar{q}\|_{\mathcal{M}(\Omega)}.
\]

Adding these two inequalities results in

\[
\langle \bar{q}_h - \bar{q}, \bar{z} - \bar{z}_h \rangle \geq 0.
\]

We introduce \( \bar{z}_h = z_h(\bar{q}) \in V_h \) defined by

\[
(\nabla v_h, \nabla \bar{z}_h) = (\bar{u}_h - u_d, v_h) \quad \text{for all} \quad v_h \in V_h,
\]

where \( \bar{u}_h = u_h(\bar{q}) \) as before. There holds:

\[
0 \leq \langle \bar{q}_h - \bar{q}, \bar{z} - \bar{z}_h \rangle = \langle \bar{q}_h - \bar{q}, \bar{z} - \bar{z}_h \rangle + \langle \bar{q}_h - \bar{q}, \bar{z}_h - \bar{z}_h \rangle
\]

\[
= \langle \bar{q}_h - \bar{q}, \bar{z} - \bar{z}_h \rangle + \langle \nabla (\bar{u}_h - \bar{u}_h), \nabla (\bar{z}_h - \bar{z}_h) \rangle
\]

\[
= \langle \bar{q}_h - \bar{q}, \bar{z} - \bar{z}_h \rangle - \|\bar{u}_h - \bar{u}_h\|^2_{L^2(\Omega)}
\]

and therefore

\[
\|\bar{u}_h - \bar{u}_h\|^2_{L^2(\Omega)} \leq c\|\bar{z} - \bar{z}_h\|_{L^\infty(\Omega)}
\]

(5.1)

since \( \|\bar{q}\|_{\mathcal{M}(\Omega)} \) and \( \|\bar{q}_h\|_{\mathcal{M}(\Omega)} \) are bounded. Note, that it now suffices to estimate the term on the right in (5.1) to obtain the final result, since

\[
\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \leq c h \|\bar{q}\|_{H^{-1}(\Omega)} + \|\bar{u}_h - \bar{u}_h\|_{L^2(\Omega)},
\]

holds with the estimate for \( \bar{u} - \bar{u}_h \) from Lemma 3.4 with \( p = 2 \).

To this end, we introduce \( \bar{z}_h \in V_h \) as the Ritz projection of \( \bar{z} \) determined by

\[
(\nabla v_h, \nabla \bar{z}_h) = (\bar{u} - u_d, v_h) \quad \text{for all} \quad v_h \in V_h,
\]

and split the last term in (5.1) as

\[
\|\bar{z} - \bar{z}_h\|_{L^\infty(\Omega)} \leq \|\bar{z} - \bar{z}_h\|_{L^\infty(\Omega)} + \|\bar{z}_h - \bar{z}_h\|_{L^\infty(\Omega)}.
\]

Using the \( L^\infty \)-estimate from [16] we obtain for the first term

\[
\|\bar{z} - \bar{z}_h\|_{L^\infty(\Omega)} \leq c h^2 |\ln h|^{\frac{1}{2}} \|\bar{u} - u_d\|_{L^\infty(\Omega)}.
\]

For the second term, define \( g_h \) to be the Ritz projection of the Green’s function at \( x_0 \in \Omega \), which fulfills

\[
(\nabla g_h, \nabla v_h) = \langle \delta_{x_0}, v_h \rangle = v_h(x_0) \quad \text{for all} \quad v_h \in V_h.
\]
Then, for the error \( e_h = \hat{z}_h - \tilde{z}_h \in V_h \) pointwise at \( x_0 \) we obtain
\[
e_h(x_0) = (\delta_{x_0}, e_h) = (\nabla g_h, \nabla e_h) = (\bar{u} - \tilde{u}_h, g_h) \leq \| \bar{u} - \tilde{u}_h \|_{L^2(\Omega)} \| g_h \|_{L^\infty(\Omega)},
\]
with Hölder’s inequality. The last term is bounded by
\[
\| g_h \|_{L^\infty(\Omega)} \leq c |\ln h|,
\]
with Lemma 3.5 applied to \( q = \delta_{x_0} \). The estimate for \( \bar{u} - \tilde{u}_h \) in \( L^2(\Omega) \) is obtained by interpolation, recalling that
\[
\| \bar{u} - \tilde{u}_h \|_{L^1(\Omega)} \leq c h^2 |\ln h|^{\frac{1}{p}} \| q \|_{L^p(\Omega)} \quad \text{and} \quad \| \bar{u} - \tilde{u}_h \|_{L^p(\Omega)} \leq c h \| q \|_{W^{1,p}(\Omega)},
\]
with Lemmas 3.3 and 3.4. Now, interpolation between these two estimates yields an estimate in the interpolation space \( [L^2(\Omega), L^1(\Omega)]_\theta = L^{p\theta}(\Omega) \) of the form
\[
\| \bar{u} - \tilde{u}_h \|_{L^{p\theta}(\Omega)} \leq c h^{1+\theta} |\ln h|^{\frac{1}{p}} \| q \|_{W^{1,p}(\Omega)}^{\theta} \| q \|_{L^p(\Omega)}^{1-\theta}.
\]
(5.2)

For \( p_\theta = \frac{2}{3} \) we have to choose \( \theta \in (0, 1) \) as
\[
\frac{2}{3} = \theta + \frac{1-\theta}{p}, \quad \text{or equivalently} \quad \theta = \frac{\frac{2}{3} - \frac{1}{p}}{1 - \frac{1}{p}}.
\]
Putting everything together, we obtain the second part of the result. The first statement is simply the special case for \( p = 2 \). \( \Box \)

**Remark 5.2.** In case \( u_d \) is bounded and \( \bar{q} \in W^{-1,p}(\Omega) \) for all \( p < \infty \), this results in a rate for the state error of (almost) \( O(h^\frac{2}{3}) \).

6. **Computational Aspects.** For the numerical computation of optimal controls we are going to consider a Tikhonov regularized version of the optimal control problem. Then, the Tichonov-parameter is driven to zero with a continuation method. The regularized problem is given in the continuous setting by
\[
\min_{q \in L^2(\Omega)} \frac{1}{2} \| u - u_d \|^2_{L^2(\Omega)} + \alpha \| q \|_{L^1(\Omega)} + \frac{\varepsilon}{2} \| q \|^2_{L^2(\Omega)} \quad \text{s.t.} \quad (\nabla u, \nabla v) = (q, v) \quad \text{for all} \ v \in V,
\]
(6.1)

where \( \varepsilon \geq 0 \) is the regularization parameter. See [10] for a detailed analysis of the connection of (6.1) and the original problem. Specifically it is shown there, that the optimal controls \( \bar{q}_\varepsilon \) converge to \( \bar{q} \) weakly in \( H^{-2}(\Omega) \) for \( \varepsilon \to 0 \). For analysis of the problem (6.1) for a fixed \( \varepsilon \) see also [18] and [9].

The optimality condition for (6.1) with \( \varepsilon > 0 \) is known to be given by the projection formula
\[
\bar{q}_\varepsilon = \frac{1}{\varepsilon} \text{sh}_\alpha(-\tilde{z}_\varepsilon),
\]
where the Nemizkij-operator \( \text{sh}_\alpha \) (soft-shrinkage) can be written as
\[
\text{sh}_\alpha(y) = \max(0, y - \alpha) - \max(0, \alpha - y),
\]
and \( \tilde{z}_\varepsilon \) fulfills the adjoint equation (2.1) with a corresponding state solution \( \bar{u}_\varepsilon \) solving (1.2). Thus, the control variable can be eliminated to obtain the system
\[
G(z, u) = \left( \begin{array}{c}
u - \Delta z - u_d \\ -\Delta u - \frac{1}{\varepsilon} \text{sh}_\beta(-z)
\end{array} \right) = 0,
\]
which can be solved with a semismooth Newton method, see e.g. [19].

We proceed completely analogously for the discrete problem. However, since the controls are discretized as nodal Dirac measures, it is not immediately clear how to interpret the regularization term in the discrete setting. For simplicity, we implement the regularization term as

$$
\frac{\varepsilon}{2} \| q_h \|_{L^2_h}^2 = \frac{\varepsilon}{2} \sum_{i=1}^{N} d_i^{-1} q_i^2
$$

(6.2)

where $q_i$ is the coefficient of the control $q_h \in \mathcal{M}_h$ at the nodal Dirac measure $\delta_{x_i}$ and $(d_i)_{i=1,...N}$ is the diagonal of the lumped mass matrix. The discrete regularized problem is then given by

$$
\min_{q_h \in \mathcal{M}_h} \frac{1}{2} \| u_h - u_d \|_{L^2(\Omega)}^2 + \alpha \| q_h \|_{M(\Omega)} + \frac{\varepsilon}{2} \| q_h \|_{L^2_h}^2
$$

s.t. $(\nabla u_h, \nabla v_h) = (q_h, v_h)$ for all $v_h \in V_h$.

(6.3)

A related mass lumping for discretization of $L^1$-control-costs is also employed in [8].

The optimality system for (6.3) can then be derived as in the continuous setting. We only point out, that here we obtain the optimality condition

$$
d_i^{-1} q_i = \frac{\varepsilon}{\alpha} s h_a (-\bar{z}_{h,\varepsilon}(x_i)) \quad \text{for} \ i = 1\ldots N,
$$

where $q_i$ is the coefficient of the optimal control $q_{h,\varepsilon} \in \mathcal{M}_h$ at the nodal Dirac $\delta_{x_i}$. The corresponding algorithm for the discrete regularized problem (6.3) was implemented with [17], and the arising linear systems were solved with a Schur-complement method and conjugate gradients.

7. Numerical examples. We present some examples to verify the rates of convergence established in Sections 4 and 5.

7.1. Example for $d = 2$. We take $\Omega = B_1(0)$ as the unit ball and construct a radially symmetric example with the optimal state given as

$$
\bar{u}(x) = -\frac{1}{2\pi} \ln(\max\{\rho, |x|\}),
$$

with a kink in the radial direction at $\rho \in [0,1)$. See Figure 7.1 for the representative cases $\rho = \frac{1}{2}$ and $\rho = 0$. For $\rho = 0$ the state $\bar{u}$ is simply a Green’s function, and the optimal control is then given by $\bar{q} = \delta_0$. For $\rho > 0$ we obtain the surface measure (given in terms of the 1-dimensional Hausdorff measure $\mathcal{H}$)

$$
\bar{q} = \frac{1}{2\pi \rho} \mathcal{H}^1 |_{\partial B_{\rho}(0)}
$$

which, due to the choice of scaling, has a norm of $\| \bar{q} \|_{M(\Omega)} = 1$. The optimal dual state can then be chosen as any element in $H^2(\Omega) \cap H_0^1(\Omega)$, such that $|\bar{z}| \leq \alpha$ and $\bar{z}|_{\partial B_{\rho}(0)} = -\alpha$. We make the specific choice

$$
\bar{z}(x) = h(|x|),
$$

where $h \in C^4([0,1])$ is a piecewise cubic polynomial interpolating $h(0) = h(1) = 0$, $h(\rho) = -\alpha$ with the choices $h'(\rho) = h'(0) = h'(1) = 0$. This yields $\bar{z} \in C^1(\Omega)$, which
is piecewise twice continuously differentiable with bounded second derivatives, and a matching desired state \( u_d \in L^\infty(\Omega) \) can be computed in strong formulation as

\[
u_d = \Delta \bar{z} + \bar{u},
\]

as depicted in Figure 7.1 for \( \rho \in \{0, \frac{1}{2}\} \). For the convenience of the reader, the exact formula for \( u_d \) is given by

\[
u_d(r) = \begin{cases} 
\alpha \frac{6(3r^2-2\rho)}{\rho^3} - \frac{1}{2\pi} \ln(\rho) & \text{for } r < \rho \\
\alpha \frac{6(3r^2-2r^2+2r+\rho)}{(\rho-1)^3r} - \frac{1}{2\pi} \ln(r) & \text{for } r \geq \rho,
\end{cases}
\]

where \( r = |x| \).

The convergence rates for a choice of \( \rho = \frac{1}{2} \) and \( \rho = 0 \) are given in Figure 7.2. The initial grid (refinement level 0) consists of five cells, a small square in the middle and four additional trapezoids at each edge, glued together at the corners. For both examples we plot the error in the cost functional \( J(\bar{q}, \bar{u}) - J(q_h, u_h) \) and the \( L^2 \)-error in the state variable. The dashed lines indicate the orders of convergence \( O(h^2) \) and \( O(h) \), which are what theory predicts for the respective quantities (up to logarithmic contributions). Since the regularization is present in the numerical computations, we also report the size of the term \( \frac{\epsilon}{2} \|q_h\|_{L^2}^2 \). As a parameter choice rule, at each refinement level the regularization parameter \( \epsilon \) is decreased until

\[
\frac{\epsilon}{2} \|q_h\|_{L^2}^2 \leq c_{\text{reg}} h^2
\]

is fulfilled, where \( c_{\text{reg}} > 0 \) is a constant chosen heuristically in advance. This is done to ensure that at least the asymptotic best case convergence behaviour of the
functional $O(|\ln h| h^2)$ should not be altered by the regularization. In Figure 7.2(a) we e.g. observe that the regularization term is an order of a magnitude smaller than the exact functional error, such that the reported error in the functional should be at least accurate in the first significant digit.

We see that the observed rates seem to agree with the rates predicted by theory. In Figure 7.2(a) the rates seem to be even slightly better, however, this is far from conclusive. In Figure 7.2(b), even though the rate for the functional is somewhat wiggly, we observe the expected rates. The wiggles could be caused by the fact that the initial mesh was perturbed slightly, and thus the approximation quality depends for a large part on the smallest distance of a grid-point to the origin, where the optimal control $\bar{q} = \delta_0$ is located. If we choose a mesh which has a point at the origin, the exact control is representable at each level, and the wiggles disappear. In the Dirac case, due to the low regularity of $u_d$, it is also clear that the rate of almost $O(h)$ for the state error is the best theoretically possible.

7.2. Example for $d = 3$. The construction of an example in three dimensions is completely analogous, except for the different Green’s function

$$\bar{u}(x) = \frac{1}{4\pi} \left( \frac{1}{\max\{\rho, |x|\}} - 1 \right),$$

thus we omit a detailed description. The final formula for $u_d$ in this case is given by

$$u_d(r) = \begin{cases} \alpha \frac{6(4r-3\rho)}{\rho^3} + \frac{1}{4\pi} \left( \frac{1}{r} - 1 \right) & \text{for } r < \rho \\ \alpha \frac{6(4r^2-3\rho r-3r+2\rho)}{(\rho-1)^3r} + \frac{1}{4\pi} \left( \frac{1}{r} - 1 \right) & \text{for } r \geq \rho, \end{cases}$$

where $r = |x|$. The computational results can be seen in Figure 7.3. Note that the parameter choice rule for $\varepsilon$ is simply the same as before. In this case, the general theory predicts an order of convergence close to $O(h)$ for the functional and close to $O(h^{1/2})$ for the $L^2$-error of the state. This is clearly observed in the case $\rho = 0$, where the optimal control $\bar{q}$ is a single Dirac delta function, see Figure 7.3(b). In this case
the rate for the state error is again the theoretically best possible. However, in the case $\rho = \frac{1}{2}$, depicted in 7.3(a), where the optimal control is a surface measure and the optimal state is Lipschitz continuous, the rates are clearly better than that. For visual comparison we plot the rates $O(h)$ and $O(h^2)$, which seem to be the closest match. With the result of Theorem 5.1 we can show an order of convergence of at least $|\ln h|^{\frac{15}{4}} h^{\frac{5}{6}}$ for the error in the state in this example, since the optimal control is an element of $W^{-1,\infty}(\Omega)$ in this case.

REFERENCES