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Nonlinear Optimization: Advanced

Exercise Sheet 1

Exercise 1.1 (Repetition: Existence of Optimal Solutions):
Let $X \subset \mathbb{R}^n$ be a nonempty set and let $f : X \to \mathbb{R}$ be a continuous objective function. We consider the constrained optimization problem

$$\min_x \ f(x) \quad \text{s.t.} \quad x \in X. \tag{P$_1$}$$

Decide whether the optimization problem (P$_1$) admits a (global) optimal solution under either of the following conditions (give a short proof or counterexample):

a) X is closed and it holds $\lim_{x\in X,\, \|x\|\to\infty} \ f(x) = \infty$.

b) X is open and it holds $\lim_{x\in X,\, \|x\|\to\infty} \ f(x) = \infty$.

c) X is closed and bounded and it holds $\lim_{x\in X,\, \|x\|\to\infty} \ f(x) = -\infty$.

d) The set X and the function f are convex.

Exercise 1.2 (Optimality Conditions for Convex Problems):
Let us consider the convex minimization problem

$$\min_x \ f(x) \quad \text{s.t.} \quad x \in X, \tag{P$_2$}$$

where $X \subseteq \mathbb{R}^n$ is a nonempty, convex set and $f : X \to \mathbb{R}$ is a convex function. Furthermore, let $\bar{x} \in X$ and assume that f is differentiable on an open neighbourhood $U \supseteq X$ of the set X.

a) Show that $\bar{x}$ is a solution of problem (P$_2$) if and only if the following conditions are satisfied:

$$\bar{x} \in X \quad \text{and} \quad \nabla f(\bar{x})^\top (x - \bar{x}) \geq 0 \quad \forall\, x \in X. \tag{VI}$$

Remark: Inequalities of the form (VI) are called *variational inequalities*.

In the following, we additionally assume that the set X is closed.

b) Let $y \in \mathbb{R}^n$ be given and consider the following special case of problem (P$_2$):

$$\min_x \ \frac{1}{2}\|x - y\|_2^2 \quad \text{s.t.} \quad x \in X. \tag{P$_3$}$$

Show that for arbitrary $y \in \mathbb{R}^n$ the minimum of (P$_3$) is uniquely attained.

c) Let $\tau > 0$ be arbitrary. Use part a) and show that $\bar{x}$ is a solution of problem (P$_2$) if and only if the following fixed point equation and optimality condition is satisfied

$$d_\tau(\bar{x}) := \bar{x} - \mathcal{P}_X(\bar{x} - \tau \nabla f(\bar{x})) = 0.$$

Remark: The Euclidean projection $\mathcal{P}_X(y)$ of $y \in \mathbb{R}^n$ onto the set X is defined as the unique minimizer of problem (P$_3$), i.e., it holds

$$\mathcal{P}_X : \mathbb{R}^n \to \mathbb{R}^n, \quad \mathcal{P}_X(y) := \operatorname*{arg\,min}_{x\in X} \ \frac{1}{2}\|x - y\|_2^2.$$

Exercise 1.3 (KKT Conditions):
In this exercise we consider an optimization problem with equality constraints

$$\min_{x\in\mathbb{R}^2} \frac{1}{2}\left(x_1^2+x_2^2\right) \qquad \text{s.t.} \qquad 1-x_1+x_2=0. \tag{P$_4$}$$

Let f denote the objective function of (P$_4$) and define $h:\mathbb{R}^2\to\mathbb{R}$, $h(x):=1-x_1+x_2$. Moreover, let $X:=\{x\in\mathbb{R}^2 : h(x)=0\}$ denote the feasible set associated with problem (P$_4$).

a) Draw the contour lines of f and the feasible set X and determine the solution $\bar{x}\in X$ of (P$_4$) graphically.

b) Modify your sketch and draw the gradients $\nabla f(\bar{x})$ and $\nabla h(\bar{x})$ (with initial point $\bar{x}$). What kind of connection exists between those two vectors?

c) Adjust the function h such that the mathematical relation between $\nabla f(\bar{x})$ and $\nabla h(\bar{x})$ from part b) changes its sign, but the feasible set X remains unchanged.

Exercise 1.4 (Geometric Properties of Constraints):
Let $g:\mathbb{R}^n\to\mathbb{R}$ be a continuously differentiable function and let $\bar{x}\in\mathbb{R}^n$ be a given point that satisfies $g(\bar{x})=0$ and $\nabla g(\bar{x})\neq 0$. Moreover, let us define $H:=\{x\in\mathbb{R}^n : g(x)=0\}$.

a) Show that the gradient $\nabla g(\bar{x})$ is perpendicular to the level set H. More specifically, show that it holds

$$\nabla g(\bar{x}) \perp \gamma'(0)$$

for every continuously differentiable curve $\gamma : I\to\mathbb{R}^n$, $I\subset\mathbb{R}$, with $\gamma(I)\subset H$ and $\gamma(0)=\bar{x}$.

b) Verify that $\bar{x}$ lies on the boundary of the set $X=\{x\in\mathbb{R}^n : g(x)\leq 0\}$ and that the gradient $\nabla g(\bar{x})$ points out of X (i.e., it holds $\bar{x}+t\nabla g(\bar{x})\notin X$ for $t>0$ sufficiently small).

c) Visualize the results from part a) and b) by drawing a sketch for the example $g:\mathbb{R}^2\to\mathbb{R}$, $g(x):=x_2-x_1^3$ and $\bar{x}=(1,1)^\top$. Draw $\bar{x}$, the sets H, X, and the vector $\nabla g(\bar{x})$ at the point $\bar{x}$.

The exercises will be discussed from **Oct, 26th** to **Nov, 3rd**.