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Nonlinear Optimization: Advanced

Exercise Sheet 2

Exercise 2.1 (Properties of the Tangent Cone):
Let $X \subset \mathbb{R}^n$ be a nonempty set and let $x \in X$ be given.

a) Show that the tangent cone $T(X, x)$ is a closed, nonempty cone.

b) Let X be additionally convex. Show that the tangent cone $T(X, x)$ can then be represented as follows
$$T(X, x) = \mathrm{cl}(\{d \in \mathbb{R}^n : \exists\ \eta > 0,\ y \in X : d = \eta(y - x)\}),$$
where $\mathrm{cl}(C)$ denotes the closure of a set C. Verify that $T(X, x)$ is also a convex set in this case.

Exercise 2.2 (Tangent Cones and Linear Constraints):

a) Let us consider the following constrained optimization problem with linear constraints
$$\min_{x\in\mathbb{R}^n} f(x) \qquad \text{s.t.} \qquad Ax = b, \quad Cx \le d, \tag{P$_1$}$$
where $f : \mathbb{R}^n \to \mathbb{R}$, $A \in \mathbb{R}^{p\times n}$, $C \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^p$, and $d \in \mathbb{R}^m$. Furthermore, let us define
$$\hat{g}(x) := Cx - d \quad \text{and} \quad \hat{h}(x) := Ax - b.$$
The feasible set of problem (P$_1$) is given by $Z := \{x \in \mathbb{R}^n : \ \hat{g}(x) \le 0,\ \hat{h}(x) = 0\}$.

Let $\bar{x} \in Z$ be arbitrary. Show that the inclusion $T_l(\hat{g}, \hat{h}, \bar{x}) \subset T(Z, \bar{x})$ is satisfied and infer that this implies $T_l(\hat{g}, \hat{h}, \bar{x}) = T(Z, \bar{x})$.

b) Now, let us consider a general constrained optimization problem of the form
$$\min_{x\in\mathbb{R}^n} f(x) \qquad \text{s.t.} \qquad g(x) \le 0, \quad h(x) = 0, \tag{P$_2$}$$
where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, and $h : \mathbb{R}^n \to \mathbb{R}^p$ are assumed to be continuously differentiable functions.

Let $\bar{x}$ be an arbitrary feasible point of problem (P$_2$) and consider the following modified problem
$$\min_{x\in\mathbb{R}^n} f(x) \qquad \text{s.t.} \qquad g^l(x) \le 0, \quad h^l(x) = 0, \tag{P$_3$}$$
with the linearized constraints $g^l(x) := g(\bar{x}) + \nabla g(\bar{x})^\top (x - \bar{x})$, $h^l(x) := h(\bar{x}) + \nabla h(\bar{x})^\top (x - \bar{x})$ and the corresponding feasible set $X_l(\bar{x}) := \{x \in \mathbb{R}^n : \ g^l(x) \le 0,\ h^l(x) = 0\}$.

Show that the two linearized tangent cones $T_l(g, h, \bar{x})$ and $T_l(g^l, h^l, \bar{x})$ coincide.

c) Infer that for any feasible point $\bar{x}$ the linearized tangent cone $T_l(g, h, \bar{x})$ of problem (P$_2$) is equal to the tangent cone $T(X_l(\bar{x}), \bar{x})$ of the linearized problem (P$_3$).

Exercise 2.3 (Abadie and Guignard Constraint Qualification):
In this exercise, we investigate different tangent cones of feasible sets of the form

$$X := \{x \in \mathbb{R}^2 : g(x) \leq 0\}.$$

Compute the tangent cone $T(X, \bar{x})$ and the corresponding linearized tangent cone $T_l(g, \bar{x})$ at the given point $\bar{x}$ and check if the two cones coincide $T(X, \bar{x}) = T_l(g, \bar{x})$. Furthermore, examine whether the Guignard constraint qualification, $T(X, \bar{x})^\circ = T_l(g, \bar{x})^\circ$, is satisfied.

a) $g(x) = (-x_1, x_2^2)^\top$, $\bar{x} = (0, 0)^\top$.

b) $g(x) = (x_2 - x_1^3, -x_2)^\top$, $\bar{x} = (0, 0)^\top$.

c) $g(x) = (-x_1, -x_2, x_1 x_2)^\top$, $\bar{x} = (0, 0)^\top$.

Hint: Use the inclusion $T(X, \bar{x}) \subset T_l(g, \bar{x})$.

The exercises will be discussed from **Nov, 9th** to **Nov, 17th**.