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Nonlinear Optimization: Advanced

Exercise Sheet 3

Exercise 3.1 (Slater's Condition):
We consider the constrained optimization problem

$$\min_{x\in\mathbb{R}^n} f(x) \quad \text{s.t.} \quad g(x) \le 0, \quad h(x) = 0, \tag{P$_1$}$$

where f, $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, and $h : \mathbb{R}^n \to \mathbb{R}^p$ are continuously differentiable functions. In addition, let us assume that each g_i, $i = 1, \ldots, m$ is convex and h is an affine, linear function.

We say that *Slater's Condition* is satisfied if there exists a point $y \in \mathbb{R}^n$ such that $g_i(y) < 0$ for all $i = 1, \ldots, m$ and $h(y) = 0$.

Show that Slater's Condition is a constraint qualification for *every* feasible point of the minimization problem (P$_1$).

Hint: Prove that Slater's Condition implies the MFCQ.

Exercise 3.2 (KKT Conditions):
In this exercise, we consider the problem

$$\begin{aligned} &\min_{x\in\mathbb{R}^n} \ (2 - x_1)^2 + x_2^4 \\ &\text{s.t.} \quad (x_1 + 1)^2 + x_2 \le 4, \quad x_1^2 - x_2 \le 1, \quad 2x_1 + x_2 \le 2. \end{aligned} \tag{P$_2$}$$

Furthermore, let us set $\bar{x} = (1, 0)^\top$.

a) Show that the MFCQ is satisfied at $\bar{x}$ and prove that $\bar{x}$ is not a regular point, i.e., the LICQ is violated.

b) Derive the KKT conditions for problem (P$_2$) and calculate the set of all multipliers $\bar{\lambda} \in \mathbb{R}^3$ such that $(\bar{x}, \bar{\lambda})$ is a KKT point of (P$_2$).

Exercise 3.3 (Projection onto a Subspace):
Let $A \in \mathbb{R}^{n\times p}$, $p \le n$ be a matrix with full column rank, i.e., $\operatorname{rank}(A) = p$, and let $y \in \mathbb{R}^n$. We consider the following problem

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|x - y\|_2^2 \quad \text{s.t.} \quad A^\top x = 0. \tag{P$_3$}$$

a) Show that the GCQ is satisfied at every feasible point.

b) Let $\bar{x} = (I - A(A^\top A)^{-1}A^\top)y$ be given. Compute a Lagrange multiplier $\bar{\mu} \in \mathbb{R}^p$ such that $(\bar{x}, \bar{\mu})$ is a KKT point of problem (P$_3$).

c) Verify that $\bar{x}$ is the unique solution of the minimization problem (P$_3$).

Exercise 3.4 (Stability of the Mangasarian-Fromovitz Constraint Qualification):
Let us consider the optimization problem

$$\min_{x\in\mathbb{R}^n} f(x) \quad \text{s.t.} \quad x \in X := \{x \in \mathbb{R}^n : g(x) \le 0,\ h(x) = 0\}, \tag{P$_4$}$$

where f, $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, m$, and $h : \mathbb{R}^n \to \mathbb{R}^p$ are continuously differentiable functions.

a) Suppose that the MFCQ is satisfied at $\bar{x} \in X$. Show that there exists a neighborhood U of $\bar{x}$ such that the MFCQ holds at all $x \in U \cap X$. You can follow the steps below:

   (i) At first, prove that there exists $\delta_1 > 0$ such that $\nabla h(x)$ has full column rank for all $x \in B_{\delta_1}(\bar{x}) := \{x \in \mathbb{R}^n : \|x - \bar{x}\|_2 < \delta_1\}$.

   (ii) Next, show that there exists $0 < \delta_2 \le \delta_1$ such that $\mathcal{A}(x) \subseteq \mathcal{A}(\bar{x})$ for all $x \in B_{\delta_2}(\bar{x}) \cap X$.

   (iii) Let $x \in B_{\delta_1}(\bar{x})$ and $d \in \mathbb{R}^n$ be arbitrary. Use exercise 3.3 and construct an *appropriate* vector $d_x \in \mathbb{R}^n$ that satisfies $\nabla h(x)^\top d_x = 0$ and

$$\nabla g_i(x)^\top d_x \le \nabla g_i(x)^\top d + \|\nabla g_i(x)\| \|\nabla h(x)(\nabla h(x)^\top \nabla h(x))^{-1} \nabla h(x)^\top d\|, \quad \forall\, i.$$

   (iv) Finally, show that there exists $0 < \delta_3 \le \delta_2$ such that the conditions *a)* and *b)* of the MFCQ (Definition 2.20) are fulfilled for all $x \in B_{\delta_3}(\bar{x}) \cap X$ and all corresponding d_x and infer that the MFCQ holds at all points $x \in B_{\delta_3}(\bar{x}) \cap X$.

b) Consider the following feasible set

$$X = \{x \in \mathbb{R}^3 : -x_1^3 - x_2 \le 0,\ -x_1^3 + x_2 \le 0,\ -x_1 - x_3^2 \le 0\}$$

   and check that the ACQ holds at $\bar{x} = (0, 0, 0)^\top$. Show that every neighborhood of $\bar{x}$ contains at least one feasible point $x \in X$ at which the ACQ is violated.

Exercise 3.5 (Connections between CQs and the Set of Lagrange Multipliers):
We consider the nonlinear program

$$\min_{x} f(x) \qquad \text{s.t.} \qquad x \in X := \{x \in \mathbb{R}^n : g(x) \le 0,\ h(x) = 0\}, \tag{P$_5$}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are continuously differentiable functions. Suppose that $\bar{x}$ is a KKT-point of problem (P$_5$) and define the set of Lagrange multipliers

$$\mathcal{M}(\bar{x}) := \{(\lambda, \mu) \in \mathbb{R}^m_+ \times \mathbb{R}^p : \text{the triple } (\bar{x}, \lambda, \mu) \text{ satisfies the KKT conditions for (P}_5)\}.$$

a) Show that the set $\mathcal{M}(\bar{x})$ is closed and convex.

b) Let us define the set of KKT-points $\mathcal{K} := \{x \in \mathbb{R}^n : x \text{ is a KKT point of (P}_5)\}$ and suppose that the MFCQ holds at $\bar{x}$. Infer that there exists a neighborhood U of $\bar{x}$ such that $\mathcal{M}(x)$ is uniformly bounded on the set $U \cap \mathcal{K}$.

c) Assume that the LICQ is satisfied at $\bar{x}$. Prove that the set $\mathcal{M}(\bar{x})$ reduces to a singleton.

The exercises will be discussed from **Nov, 23th** to **Dec, 1st**.