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Winter Semester 2016/17

Nonlinear Optimization: Advanced

Exercise Sheet 4

Exercise 4.1 (First-Order Sufficient Conditions and the Critical Cone):
We consider the general, constrained optimization problem

$$\min_{x\in\mathbb{R}^n} f(x) \qquad \text{s.t.} \qquad g(x) \le 0, \quad h(x) = 0, \tag{P$_1$}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are continuously differentiable functions. Furthermore, let $\bar{x} \in \mathbb{R}^n$ and the multipliers $\bar{\lambda} \in \mathbb{R}^m$, $\bar{\mu} \in \mathbb{R}^p$ satisfy the KKT conditions and let $X \subset \mathbb{R}^n$ denote the feasible set associated with problem (P$_1$).

a) Suppose that $\bar{x}$ satisfies the following first-order sufficient optimality conditions

$$\nabla f(\bar{x})^\top d > 0, \quad \forall\, d \in T(X, \bar{x})\backslash\{0\}.$$

Show that $\bar{x}$ is a strict local solution of the problem (P$_1$).

b) Verify the following equality: $T_+(g, h, \bar{x}, \bar{\lambda}) = \left\{ d \in T_l(g, h, \bar{x}) : \nabla f(\bar{x})^\top d = 0 \right\}$.

c) Let $\bar{x}$ be a local and unconstrained minimizer of f and assume that the strict complementarity condition is satisfied at $(\bar{x}, \bar{\lambda}, \bar{\mu})$. Infer that, in this case, it holds $T_a(g, h, \bar{x}) = T_l(g, h, \bar{x})$.

Exercise 4.2 (Second-Order Optimality Conditions):
Let $n \in \mathbb{N}$ be given. Let us consider the problem

$$\min_{x\in\mathbb{R}^n} f(x) = \sum_{j=1}^{n} x_j^j \qquad \text{s.t.} \qquad g(x) = 1 - \|x\|_2^2 \le 0, \tag{P$_2$}$$

and let $X := \{x \in \mathbb{R}^n : g(x) \le 0\}$ denote the corresponding feasible set.

a) Use an appropriate CQ and show that the ACQ is satisfied at every feasible point of (P$_2$).

b) Verify that the point $\bar{x} = (1, 0, 0, \ldots, 0)^\top \in \mathbb{R}^n$ and the multiplier $\bar{\lambda} = \frac{1}{2}$ form a KKT pair of problem (P$_2$).

c) Compute the cones $T(X, \bar{x})$ and $T_+(g, \bar{x}, \bar{\lambda})$ and simplify as far as possible.

d) Apply the second-order necessary and sufficient conditions and show that $\bar{x}$ is a local solution of problem (P$_2$) if and only if $n \le 2$.

Exercise 4.3 ((CQ2) and Second-Order Necessary Conditions):
We consider the optimization problem

$$\min_{x\in\mathbb{R}^2} f(x) = \frac{1}{2}(x_1^2 + x_2^2) \qquad \text{s.t.} \qquad g(x) = x_1^4 - x_2^2 \leq 0.$$

In the following, you can use without proof that a constraint qualification is satisfied at every feasible point $x \in \{x \in \mathbb{R}^2 : g(x) \leq 0\}$.

a) Show that $\bar{x} = (0,0)^\top$ is the unique, global solution of the above problem.

b) Let us set $\bar{\lambda} = 1$. Verify, that the pair $(\bar{x}, \bar{\lambda})$ is a KKT pair.

c) Show that the cones $T_a(g, \bar{x})$, $T_+(g, \bar{x}, \lambda)$, and $T_l(g, \bar{x})$ coincide for all $\lambda \geq 0$ and compute the cones explicitly.

d) Use Definition 2.32 and prove that the (CQ2) is violated at $(\bar{x}, \lambda)$ for all $\lambda \geq 0$.

   Hint: Show that for $d = (0,1)^\top$ there does not exist a C^2-curve γ that satisfies the properties demanded in Definition 2.32.

e) Show that $\nabla_{xx} L(\bar{x}, \bar{\lambda})$ is not positive semidefinite on $T_+(g, \bar{x}, \bar{\lambda})$ although $\bar{x}$ is a global minimizer. Why does this not contradict the second-order necessary optimality conditions?

Exercise 4.4 (Quadratic Growth and Isolated KKT-Points):
We consider the usual problem formulation

$$\min_{x\in\mathbb{R}^n} \; f(x) \qquad \text{s.t.} \qquad x \in X = \{x \in \mathbb{R}^n : g(x) \leq 0, h(x) = 0\},$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, and $h : \mathbb{R}^n \to \mathbb{R}^p$ are twice continuously differentiable functions. Let $\bar{x} \in \mathbb{R}^n$ be a KKT-point with associated multipliers $\lambda \in \mathbb{R}^m_+$ and $\mu \in \mathbb{R}^p$.

a) Suppose that the second order sufficient conditions hold at $\bar{x}$. Show that there exist $\alpha, \delta > 0$ such that the following local quadratic growth condition is satisfied:

$$f(x) - f(\bar{x}) \geq \alpha \|x - \bar{x}\|^2, \quad \forall\, x \in B_\delta(\bar{x}) \cap X.$$

   Hint: use a proof by contradiction and consider the proof of Theorem 2.29.

b) Assume that the MFCQ holds at $\bar{x}$ and that the second order sufficient conditions

$$d^\top \nabla_{xx} L(\bar{x}, \lambda, \mu) d > 0, \quad \forall\, d \in T_+(g, h, \bar{x}, \lambda) \backslash \{0\}$$

   are satisfied for every pair of Lagrange multipliers $(\lambda, \mu) \in \mathcal{M}(\bar{x})$. Show, that $\bar{x}$ is an isolated KKT-point, i.e., there exists $\delta > 0$ such that $\mathcal{M}(x) = \emptyset$ for all $x \in B_\delta(\bar{x}) \backslash \{\bar{x}\}$.

   Hint: assume that $\bar{x}$ is not an isolated KKT-point, i.e., there exists a sequence of KKT-points (x^k) with associated multipliers $(\lambda^k, \mu^k) \in \mathcal{M}(x^k)$ such that $x^k \to \bar{x}$. Use the conditions

$$\langle \lambda^k, g(x^k) \rangle = 0, \quad \langle \mu^k, h(x^k) \rangle = 0, \quad (k \in \mathbb{N})$$

   and construct an appropriate direction $d \in T_+(g, h, \bar{x}, \lambda) \backslash \{0\}$ that violates the second order optimality conditions.

The exercises will be discussed from **Dec, 8th** to **Dec, 15th**.