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Winter Semester 2016/17

Nonlinear Optimization: Advanced

Exercise Sheet 5

Exercise 5.1 (Multiple Choice Test):
The first and second multiple choice tests are available on `www.moodle.tum.de`. Take the tests and answer the different multiple choice questions online on moodle.

Exercise 5.2 (Duality and Linear Programming):
Let us consider the linear program

$$\min_{x\in\mathbb{R}^n} \ c^\top x \qquad \text{s.t.} \qquad Ax - b \le 0, \tag{P$_1$}$$

where $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and $A \in \mathbb{R}^{m\times n}$ are given vectors and matrices.

a) Compute the dual problem of (P_1) and show that the dual of the dual problem coincides with the primal problem (P_1).

b) Prove that the following statements are equivalent:

(i) The primal problem (P_1) has a solution.

(ii) The dual problem has a solution.

(iii) The feasible set of the dual problem is nonempty and the dual objective function is bounded from above on the feasible set.

(iv) The feasible set of the primal problem is nonempty and the primal objective function is bounded from below on the feasible set.

Show that any of the latter conditions also implies that there is no duality gap between the primal and dual problem, i.e., every solution $\bar{x}$ of problem (P_1) and every solution $\bar{\lambda}$ of the dual problem satisfy $p(\bar{x}) = c^\top \bar{x} = d(\bar{\lambda})$.

Hint: To prove the direction "(iv) $\implies$ (i)", show that every vector $d = (d_1^\top, d_2^\top, d_3^\top, d_4^\top)^\top \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n$ with

$$d_i \ge 0, \ i = 1, ..., 3 \quad \text{and} \quad \begin{pmatrix} A^\top d_1 \\ Ad_4 + d_2 \end{pmatrix} = \begin{pmatrix} -c \\ b \end{pmatrix} \cdot d_3$$

satisfies $b^\top d_1 + c^\top d_4 \ge 0$. (You can use a proof of contradiction). Then, apply Farkas' Lemma to establish existence of a primal solution.

c) Verify that the results in part a) and b) can also be applied to linear programs with additional linear equality constraints of the form

$$Cx = e, \quad C \in \mathbb{R}^{p\times n}, \quad e \in \mathbb{R}^p.$$

Exercise 5.3 (Trajectory of the Quadratic Penalty Method):
Consider the general constrained optimization problem

$$\min_{x\in\mathbb{R}^n} \ f(x) \qquad \text{s.t.} \qquad g(x) \le 0, \quad h(x) = 0,$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are continuously differentiable functions.

a) Suppose that f, g are convex and h is an affine function. Show that, in this case, the quadratic penalty function $P_\alpha(x) := f(x) + \frac{\alpha}{2}\sum_{i=1}^{m} (\max\{0, g_i(x)\})^2 + \frac{\alpha}{2}\sum_{i=1}^{p}(h_i(x))^2$ is convex on $\mathbb{R}^n$ for every $\alpha \ge 0$.

Now, let us consider the concrete problem

$$\min_{x\in\mathbb{R}^2} \ f(x) = x_1^2 + 4x_2 + x_2^2 \qquad \text{s.t.} \qquad g(x) = -x_2 \le 0. \tag{P$_2$}$$

In the quadratic penalty method, each iterate is determined as the global solution x^k of the unconstrained problem

$$\min_{x\in\mathbb{R}^n} \ P_{\alpha_k}(x),$$

where $(\alpha_k) \subset \mathbb{R}_{++}$ is a monotonically increasing sequence which satisfies $\lim_{k\to\infty} \alpha_k = \infty$.

b) Show that the minimum of problem (P$_2$) is uniquely attained. Calculate the global solution $\bar{x}$ of (P$_2$) explicitly and derive a corresponding Lagrange multiplier $\bar{\lambda}$.

c) Verify that the penalty function P_α, which is associated with problem (P$_2$), possesses a unique, global solution $\bar{x}_\alpha$ for all $\alpha > 0$ and compute $\bar{x}_\alpha$.

d) Show that $\bar{x} = \lim_{\alpha\to\infty} \bar{x}_\alpha$ and $\bar{\lambda} = \lim_{\alpha\to\infty} \alpha \max\{0, g(\bar{x}_\alpha)\}$. Without using part a), explain that $\bar{x}$ is a global solution of problem (P$_2$).

e) Investigate the condition number of the Hessian $\nabla^2 P_\alpha(\bar{x}_\alpha)$ for $\alpha \to \infty$.

Exercise 5.4 (Second Order Necessary Conditions Revisited *):
We consider the optimization problem

$$\min_{x} \ f(x) \qquad \text{s.t.} \qquad x \in X := \{x \in \mathbb{R}^n : g(x) \le 0,\ h(x) = 0\}, \tag{P$_3$}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, and $h : \mathbb{R}^n \to \mathbb{R}^p$ are twice continuously differentiable functions. In this exercise, we want to prove the following second order optimality conditions:

> Let $\bar{x} \in X$ be a local solution of problem (P$_3$) and suppose that the MFCQ holds at $\bar{x}$. Then, the critical cone $\mathcal{C}(\bar{x}) := T_+(g, h, \bar{x}, \lambda)$ does not depend on the multiplier $\lambda \in \mathbb{R}^m_+$ and it holds
>
> $$\max_{(\lambda,\mu)\in\mathcal{M}(\bar{x})} \ d^\top \nabla_{xx} L(\bar{x}, \lambda, \mu) d \ \ge \ 0, \quad \forall\ d \in \mathcal{C}(\bar{x}). \tag{SNC}$$

Here, $\mathcal{M}(\bar{x})$ again denotes the set of all Lagrange multipliers associated with the local solution $\bar{x}$ (see also exercise 3.5). In the following, let $d \in \mathcal{C}(\bar{x})$ be arbitrary.

a) Consider the linear program

$$\min_{w\in\mathbb{R}^n,\, z\in\mathbb{R}} \ z \qquad \text{s.t.} \qquad \begin{cases} \nabla f(\bar{x})^\top w + d^\top \nabla^2 f(\bar{x}) d \le z \\ \nabla h_i(\bar{x})^\top w + d^\top \nabla^2 h_i(\bar{x}) d = 0 & \forall\ i = 1, ..., p, \\ \nabla g_i(\bar{x})^\top w + d^\top \nabla^2 g_i(\bar{x}) d \le z & \forall\ i \in \mathcal{A}(\bar{x}), \end{cases} \tag{L}$$

and prove that its feasible set is nonempty.

b) Now, assume that there exists $w \in \mathbb{R}^n$ such that $\nabla f(\bar{x})^\top w + d^\top \nabla^2 f(\bar{x}) d < 0$ and

$$\nabla h_i(\bar{x})^\top w + d^\top \nabla^2 h_i(\bar{x}) d = 0, \ \forall\, i = 1, ..., p, \quad \nabla g_i(\bar{x})^\top w + d^\top \nabla^2 g_i(\bar{x}) d < 0, \ \forall\, i \in \mathcal{A}(\bar{x}).$$

Similar to the proof of Lemma 2.33, use the implicit function theorem and construct a twice continuously differentiable function $r : I \to \mathbb{R}^n$ and an open interval $I \subset \mathbb{R}$, $0 \in I$, such that

$$h(x(t)) = 0, \quad \forall\, t \in I, \quad \text{where} \quad x : I \to \mathbb{R}^n, \ \ x(t) := \bar{x} + td + \tfrac{1}{2}t^2 w + r(t).$$

c) Use the properties of d and w and show that the implicit function r satisfies

$$r(0) = 0, \quad r'(0) = 0, \quad r''(0) = 0, \quad r(t) = o(t^2) \quad (t \to 0).$$

Infer that the parabolic path $x(t)$ is feasible for problem (P_3) for all $t \geq 0$ sufficiently small.

d) Combine the last results and the local optimality of $\bar{x}$ to prove that the objective function of the linear program (L) is bounded below by zero on the feasible set.

e) Compute the dual problem of (L). Use exercise 5.1 and the MFCQ to finally establish the second order necessary conditions (SNC).

Hint: Use that the MFCQ is equivalent to the PLICQ and apply the PLICQ.

⋆ **Remark:** this is a special and additional Christmas exercise. ☺

The exercises will be discussed from **Dec, 21th** to **Jan, 12th**.