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Optimization with Partial Differential Equations

http://www-m1.ma.tum.de/bin/view/Lehrstuhl/MUlbrichOptPDEsWS1516

Exercise Sheet 1

Exercise 1.1 (Computation of the gradient of the reduced objective function):

Consider the *finite-dimensional optimal control problem*

$$\min_{y\in\mathbb{R}^n,\, u\in\mathbb{R}^q} J(y,u) \quad \text{s.t.} \quad e(y,u)=0 \tag{1}$$

with given functions $J \in \mathcal{C}^1(\mathbb{R}^n\times\mathbb{R}^q,\mathbb{R})$ and $e\in\mathcal{C}^1(\mathbb{R}^n\times\mathbb{R}^q,\mathbb{R}^n)$. Assume that the *state equation* $e(y,u)=0$ possesses a unique solution $y(u)$ for every *control* $u\in\mathbb{R}^q$, satisfying $e(y(u),u)=0$. Problem (1) is then equivalent to the *reduced problem*

$$\min_{u\in\mathbb{R}^q} \hat{J}(u) := J(y(u),u),$$

where $\hat{J}$ is called the *reduced objective function.* Furthermore, suppose that $e_y(y(u),u)\in\mathbb{R}^{n\times n}$ is invertible for all $u\in\mathbb{R}^q$.

a) Differentiate the equation $e(y(u),u)=0$ w. r. t. u and compute the derivative $y'(u)$ of the *control-to-state mapping* $u\mapsto y(u)$.

b) Given a direction $v\in\mathbb{R}^q$ derive—analogously to part a)—a system of linear equations for the computation of the *sensitivity* $d_v y(u) := y'(u)v$. Use the quantity $d_v y(u)$ to calculate the directional derivative $\hat{J}'(u)v$.

c) Show that
$$\nabla\hat{J}(u) = \hat{J}'(u)^T = e_u(y(u),u)^T p + \nabla_u J(y(u),u),$$
where the *adjoint state* $p=p(u)\in\mathbb{R}^n$ solves the *adjoint equation*
$$e_y(y(u),u)^T p = -\nabla_y J(y(u),u).$$

d) Compare the computational costs for computing $\nabla\hat{J}(u)$ with the *sensitivity method* from b) and with the *adjoint method* from c).

Exercise 1.2 (Generalized Hölder's inequality):

Prove the following generalization of Hölder's inequality:

Let $m,n\in\mathbb{N}$ and $\Omega\subset\mathbb{R}^n$ be measurable with positive measure. Furthermore, let the scalars $p_i\in[1,\infty]$ with $\sum_{i=1}^m \frac{1}{p_i}\le 1$ and the functions $u_i\in L^{p_i}(\Omega)$ be given for all $i\in\{1,\dots,m\}$ and define $r := \left(\sum_{i=1}^m \frac{1}{p_i}\right)^{-1}$ (with the conventions $\frac{1}{\infty}=0$ and $\frac{1}{0}=\infty$). Then it holds

$$\prod_{i=1}^m u_i \in L^r(\Omega) \quad \text{and} \quad \left\|\prod_{i=1}^m u_i\right\|_{L^r(\Omega)} \le \prod_{i=1}^m \|u_i\|_{L^{p_i}(\Omega)}.$$

Exercise 1.3 (Existence of a solution):

Let $\Omega := (0,1) \subset \mathbb{R}$ and $Y := W^{1,4}(\Omega)$. Consider the functional

$$J : Y \to \mathbb{R}, \quad J(y) := \int_\Omega \left(y'(x)^2 - 1\right)^2 + y(x)^4 \, dx.$$

Prove the following statements:

a) The functional J is continuous.

b) The optimization problem

$$\min_{y \in Y} J(y)$$

has no solution.

Hint: Show that $\inf_{y \in Y} J(y) = 0$, but that there exists no $y \in Y$ with $J(y) = 0$.

Exercise 1.4 (Differential operators and weak solutions):

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain with Lipschitz boundary $\partial\Omega$, $y \in H_0^1(\Omega)$ and $\beta \in H^1(\Omega)^n$, i. e. $\beta = (\beta_1, \ldots, \beta_n)^T : \Omega \to \mathbb{R}^n$ with $\beta_1, \ldots, \beta_n \in H^1(\Omega)$.

a) How does the space dimension $n \in \mathbb{N}$ have to be selected such that the integral

$$\int_\Omega \beta(x) \cdot \nabla y(x) \, y(x) \, dx$$

attains a finite value?

In the following, let n be selected according to the result from part a).

b) Prove that

$$\int_\Omega \beta(x) \cdot \nabla y(x) \, y(x) \, dx = -\frac{1}{2} \int_\Omega \mathrm{div}\beta(x) \, y(x)^2 \, dx.$$

c) Consider the *partial differential equation* (PDE)

$$-\Delta y + \beta \cdot \nabla y + cy = f \quad (\text{in } \Omega), \quad y = 0 \quad (\text{on } \partial\Omega) \tag{2}$$

with $f \in L^2(\Omega)$ and $c : \Omega \to \mathbb{R}$. Provide a sufficient condition for c and β such that (2) admits a unique weak solution.

This exercise sheet will be discussed in the exercise classes from **Oct 20th** to **Oct 26th**.