Technische Universität München
Center for Mathematics
Chair of Mathematical Optimization, M1
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Optimization with Partial Differential Equations

http://www-m1.ma.tum.de/bin/view/Lehrstuhl/MUlbrichOptPDEsWS1516

Exercise Sheet 2

Exercise 2.1 (Differentiability of superposition operators):

Let $f \in \mathcal{C}^1(\mathbb{R})$ be Lipschitz continuous on $\mathbb{R}$ and let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain. We consider the so-called *superposition operator*

$$F : L^p(\Omega) \to L^p(\Omega), \quad y \mapsto F(y) := f \circ y,$$

for some $p \in [1, \infty]$, i.e. we have $F(y)(x) = f(y(x))$ for all $x \in \Omega$.

Prove the following statements:

a) The operator F is well-defined and Lipschitz continuous.

b) The operator F is G-differentiable in every $y \in L^p(\Omega)$ with derivative $F'(y)v = f'(y)v$.

c) The operator F is F-differentiable for $p = \infty$.
Hint: Use that f' is uniformly continuous on compact intervals.

Provide an example that the operator F does not have to be F-differentiable for $p \in [1, \infty)$.

Exercise 2.2 (Elliptic PDEs as minimization problem):

Let Y be a Hilbert space, let $a : Y \times Y \to \mathbb{R}$ be a coercive, continuous and symmetric bilinear form and $b \in Y^*$. We consider the quadratic optimization problem

$$\min_{y \in Y} \; J(y) := \frac{1}{2} a(y, y) - \langle b, y \rangle_{Y^*,Y}. \tag{1}$$

a) Compute the Fréchet derivative of J.

b) Argue that J is strictly convex and that (1) has at most one solution.

c) Show that solving (1) is equivalent to solving the *variational equation*

$$a(y, v) = \langle b, v \rangle_{Y^*,Y} \quad \forall \, v \in Y.$$

d) Let now $\Omega \subset \mathbb{R}^n$ be an open, bounded Lipschitz domain, $Y := H_0^1(\Omega)$ and $f \in L^2(\Omega)$. Determine the bilinear form a and the functional b such that solving the minimization problem (1) is equivalent to solving the weak formulation of the PDE

$$-\Delta y = f \quad (\text{in } \Omega), \quad y = 0 \quad (\text{on } \partial\Omega).$$

Exercise 2.3 (Non-homogeneous boundary data):

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded Lipschitz domain. Consider the PDE

$$-\Delta y = f \quad (\text{in } \Omega), \quad y = g \quad (\text{on } \partial\Omega) \tag{2}$$

with right hand side $f \in L^2(\Omega)$ and boundary data $g \in L^2(\partial\Omega)$. Assume that there exists an extension $u_g \in H^1(\Omega)$ such that $\mathrm{T}u_g = g$, where $\mathrm{T} : H^1(\Omega) \to L^2(\partial\Omega)$ is the trace operator.

a) Transform the PDE (2) into a problem with homogeneous boundary data and give its weak formulation.

b) Prove that the solution that you obtain via the transformed problem does not depend on the choice of the extension u_g.

This exercise sheet will be discussed in the exercise classes from **Nov 3rd** to **Nov 9th**.