Technische Universität München
Center for Mathematics
Chair of Mathematical Optimization, M1
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Optimization with Partial Differential Equations

http://www-m1.ma.tum.de/bin/view/Lehrstuhl/MUlbrichOptPDEsWS1516

Exercise Sheet 3

Exercise 3.1 (Existence of a solution to an optimal control problem):
Let $\Omega \subset \mathbb{R}^3$ be an open, bounded Lipschitz domain. Prove the existence of a solution to the optimal control problem

$$\min_{(y,u)\in Y\times U} J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)}$$
$$\text{s.t.} \begin{cases} -\Delta y + uy = f & (\text{in } \Omega) \\ y = 0 & (\text{on } \partial\Omega) \end{cases}$$
$$a \le u \le b \quad (\text{a.e. in } \Omega)$$

with $Y = H^1_0(\Omega)$, $U = L^2(\Omega)$, $y_d,\ f \in L^2(\Omega)$, $a,\ b \in L^2(\Omega)$ with $0 \le a \le b$ a.e. and $\alpha \in \mathbb{R}_{>0}$.

Exercise 3.2 (Fréchet derivative and Riesz representation):
Let $\Omega := (0,1) \subset \mathbb{R}$ and

$$J(y) := \int_\Omega \cos(\pi x) y^3(x)\, dx.$$

a) Show that $J : H^1(\Omega) \to \mathbb{R}$ is well-defined.

b) Prove that J is F-differentiable with derivative $J'(y) \in H^1(\Omega)^*$,

$$\langle J'(y), v\rangle_{H^1(\Omega)^*,H^1(\Omega)} = \int_\Omega 3\cos(\pi x) y^2(x) v(x)\, dx \quad \forall v \in H^1(\Omega).$$

c) Consider the unique Riesz representation $\nabla J(y) \in H^1(\Omega)$ of $J'(y) \in H^1(\Omega)^*$,

$$(\nabla J(y), v)_{H^1(\Omega)} = \langle J'(y), v\rangle_{H^1(\Omega)^*,H^1(\Omega)} \quad \forall v \in H^1(\Omega).$$

Show that $\nabla J(y) = g$ is obtained as the weak solution of the differential equation

$$-g''(x) + g(x) = 3\cos(\pi x) y^2(x) \quad (\text{in } \Omega), \quad g'(0) = g'(1) = 0.$$

Exercise 3.3 (Adjoint calculus):

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded Lipschitz domain and let a function $\kappa \in \mathcal{C}(\overline{\Omega})$ be given with $\min_{x\in\overline{\Omega}} \kappa(x) > 0$. We consider the optimal control problem

$$\begin{aligned} \min_{y\in Y, u\in U} J(y,u) &:= \frac{1}{2}\left\|y - y_d\right\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\left\|u\right\|^2_U \\ \text{s.t.} &\begin{cases} -\operatorname{div}(\kappa\nabla y) + y = f & (\text{in } \Omega) \\ \frac{\partial y}{\partial \nu} = u & (\text{on } \partial\Omega) \end{cases} \end{aligned} \tag{1}$$

with $Y = H^1(\Omega)$, $U = L^2(\partial\Omega)$, and given $y_d, f \in L^2(\Omega)$ and $\alpha > 0$.

a) Give the weak formulation of the state equation and write it abstractly as $e(y,u) = 0$ with

$$e : Y \times U \to Y^*, \quad e(y,u) = Ay + Bu - g$$

with linear operators $A : Y \to Y^*$, $B : U \to Y^*$ and $g \in Y^*$.

b) Argue that for each $u \in U$ there exists a unique $y(u) \in Y$ fulfilling $e(y(u),u) = 0$.

c) Show that $J : Y \times U \to \mathbb{R}$ and $e : Y \times U \to Y^*$ are continuously Fréchet-differentiable. Compute the partial derivatives of J and e and argue that $e_y(y(u),u)$ is boundedly invertible.

d) Let $\hat{J}(u) := J(y(u),u)$ be the reduced objective function. Derive the adjoint equation in strong form and compute the adjoint representation of the gradient $\nabla\hat{J}(u) \in L^2(\partial\Omega)$.

This exercise sheet will be discussed in the exercise classes from **Nov 17th** to **Nov 23rd**.