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Optimization with Partial Differential Equations

http://www-m1.ma.tum.de/bin/view/Lehrstuhl/MUlbrichOptPDEsWS1516

Exercise Sheet 4

Exercise 4.1 (Computation of derivatives):

Consider again the following boundary control problem with semilinear state equation (Example 3.4 in the lecture):

$$\min_{y\in Y, u\in U} J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_U$$
$$\text{s.t.} \quad \begin{cases} -\Delta y + y^3 &= 0 \qquad (\text{in } \Omega) \\ \frac{\partial y}{\partial \nu} + y &= u \qquad (\text{on } \partial\Omega) \end{cases} \tag{1}$$
$$l \le u \le r \quad (\text{a.e. on } \partial\Omega)$$

with an open, bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ ($n \in \{2,3\}$), $Y := H^1(\Omega)$, $U := L^2(\partial\Omega)$, given $y_d \in L^2(\Omega)$, $l, r \in L^2(\partial\Omega)$ with $l \le r$ a.e. on $\partial\Omega$, $\alpha \in \mathbb{R}_{\ge 0}$, and the set of admissible controls $U_{\text{ad}} := \{u \in L^2(\partial\Omega) : l \le u \le r \text{ a.e. on } \partial\Omega\}$.
It was already shown that the weak form of the state equation $e(y,u) = 0$ with

$$e : Y \times U \to Y^*, \quad \langle e(y,u), v\rangle_{Y^*,Y} := (\nabla y, \nabla v)_{L^2(\Omega)^n} + (y^3, v)_{L^2(\Omega)} + (y-u, v)_{L^2(\partial\Omega)}$$

possesses a unique solution $y(u)$ for all $u \in U$ and that problem (1) has a solution.

a) Show that $F : L^6(\Omega) \to L^2(\Omega)$, $y \mapsto F(y) := y^3$ is continuously Fréchet-differentiable and compute the derivative.

b) Compute the partial derivatives of J and e and argue that $e_y(y,u)$ is boundedly invertible.

c) Derive the linearized state equation as well as the adjoint equation in weak and strong form.

d) Compute the adjoint representation of the gradient of the reduced objective function.

Exercise 4.2 (Finite-dimensional control of ODEs):

Consider the optimal control problem

$$\min_{y\in Y, u\in U} J(y,u) := \int_0^T g(t,y(t))\,dt + h(y(T),u) \quad \text{s.t.} \quad y' = f(t,y) \ \text{ in } [0,T], \quad y(0) = u \tag{2}$$

with $n \in \mathbb{N}$, $T \in \mathbb{R}_{>0}$, $Y := W^{1,\infty}([0,T])^n$, $U := \mathbb{R}^n$, $f \in \mathcal{C}^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^n)$, $g \in \mathcal{C}^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$, $h \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$. Assume that the state equation $y' = f(t,y), y(0) = u$ has a unique solution $y(u)$ defined on the whole interval $[0,T]$ for every initial datum $u \in U$.

a) Write down the Lagrange function $L : Y \times U \times Y \times U \to \mathbb{R}$ for problem (2). Use a multiplier $\lambda \in Y$ for the ODE constraint and another multiplier $\mu \in U$ for the initial condition.

b) Show that L is F-differentiable w. r. t. y and derive the adjoint equation for $\lambda = \lambda(u) \in Y$. For this purpose, consider its variational form

$$\langle L_y(y,u,\lambda,\mu), v\rangle_{Y^*,Y} = 0,$$

perform integration by parts, and derive an ODE for λ and an additional condition for μ by testing with suitable $v \in Y$.

c) Assume that the adjoint equation has a unique solution $\lambda(u)$ for every u. Show that L is F-differentiable w. r. t. u and derive the adjoint representation of the gradient of the reduced objective function.

d) Now let a regular matrix $A \in \mathbb{R}^{n\times n}$ and two vectors $y_T, b \in \mathbb{R}^n$ be given, and let $f(t,y) := Ay$, $g(t,y) := b^T y$, and $h(y,u) := \frac{1}{2}\|y - y_T\|_2^2$. Compute an optimal solution of problem (2).

This exercise sheet will be discussed in the exercise classes from **Dec 1st** to **Dec 7th**.