Technische Universität München
Center for Mathematics
Chair of Mathematical Optimization, M1
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Optimization with Partial Differential Equations

http://www-m1.ma.tum.de/bin/view/Lehrstuhl/MUlbrichOptPDEsWS1516

Exercise Sheet 6

Exercise 6.1 (Second-order sufficient conditions, revisited):

Let $\Omega := (1, \infty)$ and consider the unconstrained optimization problem

$$\min_{y \in Y} J(y) := \int_\Omega \frac{y(x)^2}{x} - y(x)^3 \, dx \tag{1}$$

on $Y := L^3(\Omega)$, and let $\bar{y} \equiv 0$.

a) Prove that the objective function $J : Y \to \mathbb{R}$ is well-defined and twice continuously Fréchet-differentiable and compute the first and second derivative.

b) Prove that $\bar{y}$ fulfills the first-order optimality conditions of problem (1) and write them as equation in a suitable function space.

c) Show that
$$\langle J''(\bar{y})v, v\rangle_{Y^*,Y} > 0$$
holds for all $v \in Y \backslash \{0\}$.

d) Show that $\bar{y}$ is no local solution of problem (1). Why is it not possible to apply exercise 5.1?

Exercise 6.2 (Conditional gradient method):

Let W be a reflexive Banach space, $W_{\text{ad}} \subset W$ be a nonempty, closed, bounded, convex set, and let $J : W \to \mathbb{R}$ be G-differentiable. Consider the optimization problem

$$\min_{w \in W} J(w) \quad \text{s.t.} \quad w \in W_{\text{ad}}. \tag{2}$$

The *conditional gradient method* for the iterative solution of this problem works as follows: An initial point $w^0 \in W_{\text{ad}}$ is selected. Then one repeats for $k = 0, 1, 2, \ldots$:

1. Determine $v^k \in \arg\min_{v \in W_{\text{ad}}} \langle J'(w^k), v\rangle_{W^*,W}$ and set $s^k := v^k - w^k$.
2. If $\langle J'(w^k), s^k\rangle_{W^*,W} \geq 0$ holds, then STOP the algorithm.
3. Determine $\sigma_k \in \arg\min_{\sigma \in (0,1]} J(w^k + \sigma s^k)$.
4. Set $w^{k+1} := w^k + \sigma_k s^k$ (and go to step 1).

Argue that this algorithm is well-defined and produces a (possibly finite) sequence $(w_k)_{k \in \mathbb{N}_0} \subset W_{\text{ad}}$ such that $(J(w^k))_k$ is strictly decreasing. Can you explain the stopping criterion in step 2?

Exercise 6.3 (Application of the conditional gradient method):

We want to apply the conditional gradient method from exercise 6.2 to the optimal control problem

$$\begin{aligned} \min_{(y,u)\in Y\times U} \; & J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \\ \text{s. t.} \quad & \begin{cases} -\Delta y + y = gu + f & (\text{in } \Omega), \\ \qquad\quad\; y = 0 & (\text{on } \partial\Omega), \end{cases} \qquad u \in U_{\text{ad}} \end{aligned} \tag{3}$$

with an open, bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $Y := H^1_0(\Omega)$, $U := L^2(\Omega)$, $y_d, f \in L^2(\Omega)$, $g \in L^\infty(\Omega)$, $\alpha > 0$, and $U_{\text{ad}} := \{u \in U : l \le u \le r \text{ a.e. in } \Omega\}$ with $l, r \in L^2(\Omega)$, $l \le r$ a.e. in Ω. You can use without a proof that the state equation in (3) admits a unique weak solution $y(u) \in Y$ for every $u \in U$ and fixed f and g and that the solution operator $u \mapsto y(u)$ is affine and continuous.

a) Give the reduced problem and argue that the prerequisites from exercise 6.2 hold.

b) Determine the adjoint equation in weak and strong form and argue that the adjoint state is uniquely determined.

c) Compute the adjoint-based representation of the gradient of the reduced objective function.

d) Formulate the conditional gradient method from exercise 6.2 applied to the reduced problem from part a). Note that v^k and σ_k in steps 1 and 3 can be computed explicitly for this problem.

This exercise sheet will be discussed in the exercise classes from **Jan 12th** to **Jan 18th**.