Technische Universität München
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Chair of Mathematical Optimization, M1
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Optimization with Partial Differential Equations

http://www-m1.ma.tum.de/bin/view/Lehrstuhl/MUlbrichOptPDEsWS1516

Exercise Sheet 7

Exercise 7.1 (Obstacle problem):

Given an open, bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ ($n \in \{2,3\}$), consider for $k \in \{1,2,3\}$ the elliptic obstacle problem

$$\min_{y \in Y_k} J(y) := \frac{1}{2}(\nabla y, \nabla y)_{L^2(\Omega)^n} - (f,y)_{L^2(\Omega)} \quad \text{s.t.} \quad l_k \le y \le r_k \text{ a.e. in } \Omega \tag{1}$$

with Banach spaces $Y_k \subset H_0^1(\Omega)$, $f \in L^2(\Omega)$, $l_k, r_k \in Y_k$, $l_k < r_k$ everywhere in Ω.

a) Determine a variational inequality which constitutes a first-order necessary optimality condition for problem (1) with $k = 1$, $Y_1 := H_0^1(\Omega)$.

b) Transform the box constraint $l_2 \le y \le r_2$ into a cone constraint and determine the KKT conditions for problem (1) with $k = 2$, $Y_2 := H_0^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$.

c) Having $k = 3$, $Y_3 := H_0^1(\Omega) \cap H^2(\Omega)$, show that an optimal solution $\bar{y} \in Y_3$ of problem (1) fulfills the following optimality system: There exist $\lambda, \rho \in L^2(\Omega)$ with

$$\begin{aligned} (\nabla \bar{y}, \nabla v)_{L^2(\Omega)^n} - (f,v)_{L^2(\Omega)} + (\rho, v)_{L^2(\Omega)} - (\lambda, v)_{L^2(\Omega)} &= 0 \quad \forall\, v \in Y_3 \\ \bar{y} \ge l_3,\ \lambda \ge 0,\ \lambda(\bar{y} - l_3) &= 0 \quad \text{a.e. in } \Omega \\ \bar{y} \le r_3,\ \rho \ge 0,\ \rho(r_3 - \bar{y}) &= 0 \quad \text{a.e. in } \Omega. \end{aligned}$$

Exercise 7.2 (The classical Newton's method):

Let X and Y be Banach spaces and let $G : X \to Y$ be a continuously F-differentiable operator. By $x^* \in X$ we denote a solution of the nonlinear equation $G(x) = 0$. Furthermore, let $G'(x^*)$ be continuously invertible.

Starting at $x^0 \in X$ close to x^*, the generalized Newton's method (Algorithm 6.1 from the lecture) with the special choice $M_k = G'(x^k)$ is applied, i. e. in each iteration the step s^k is obtained by solving $G'(x^k)s = -G(x^k)$ and the next iterate $x^{k+1} := x^k + s^k$ is computed.

a) For $d^k := x^k - x^*$ prove the approximation condition

$$\|G(x^* + d^k) - G(x^*) - M_k d^k\|_Y = o(\|d^k\|_X) \quad \text{for } \|d^k\|_X \to 0.$$

b) Now, let G' be Hölder continuous of order $\alpha \in (0,1]$ near x^*: It holds $\|G'(x) - G'(\tilde{x})\|_{\mathcal{L}(X,Y)} \le L\,\|x - \tilde{x}\|_X^\alpha$ for some constant $L \ge 0$ and all $x, \tilde{x} \in X$ close enough to x^*. Prove the sharper estimate

$$\|G(x^* + d^k) - G(x^*) - M_k d^k\|_Y \le \frac{L}{1+\alpha}\|d^k\|_X^{1+\alpha} = \mathcal{O}(\|d^k\|_X^{1+\alpha}) \quad \text{for } \left\|d^k\right\|_X \to 0.$$

Hint: Use the fundamental theorem of calculus.

c) What can you deduce about the local convergence order of the *classical Newton's method*?

Exercise 7.3 (A semilinear optimal control problem):

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded Lipschitz domain. Consider the optimal control problem

$$\min_{(y,u)\in Y\times U} J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{1}{2}\|y-y_b\|^2_{L^2(\partial\Omega)} + \frac{\alpha}{2}\|u\|^2_U$$
$$\text{s. t.} \begin{cases} -\Delta y + y + y^3 &= 0 \quad (\text{in } \Omega), \\ \dfrac{\partial y}{\partial \nu} + y &= u \quad (\text{on } \partial\Omega), \end{cases} \qquad u \in U_{\text{ad}} := \{u \in U : l \le u \le r \text{ a. e.}\} \tag{2}$$

with $Y := H^1(\Omega)$, $U := L^2(\partial\Omega)$, and given $y_d \in L^2(\Omega)$, $y_b \in L^2(\partial\Omega)$, $\alpha > 0$, and $l, r \in L^2(\partial\Omega)$ with $l \le r$ a. e.

You can use without a proof that for every $u \in U$ there exists a unique state $y(u) \in Y$ solving the weak form of the state equation.

a) Give the weak formulation $e(y,u) = 0$ of the state equation.

b) Compute the first partial derivates of J and e. You can use without a proof that $F : L^6(\Omega) \to L^2(\Omega), y \mapsto y^3$ is continuously Fréchet-differentiable with the derivative $F'(y)v = 3y^2v$.

c) Let $y \in H^1(\Omega)$ be given. Show that the bilinear form $a : Y \times Y \to \mathbb{R}$,

$$a(v,w) := (v,w)_{H^1(\Omega)} + (3y^2v, w)_{L^2(\Omega)} + (v,w)_{L^2(\partial\Omega)} \quad \forall\, v, w \in Y$$

is continuous. Give a short justification for all involved estimates.

d) Derive the adjoint equation in weak and strong form. You can use without a proof that $e_y(y,u)$ has a bounded inverse for all $u \in U$.

e) Compute the adjoint based representation of the gradient of the reduced objective function $\hat{J}(u) := J(y(u),u)$.

f) State a necessary optimality condition for problem (2).

This exercise sheet will be discussed in the exercise classes from **Jan 26th** to **Feb 1st**.